Compactly Supported Correlation Functions

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Abstract

This article proposes compactly supported correlation functions, which parameterize the smoothness of the associated stationary and isotropic random field. The constructions are straightforward, and compact support is relevant for various ends: computationally efficient spatial prediction, fast and exact simulation, and appeal among practicioners.

Keywords: Fractal dimension; isotropic; kriging; long-memory dependence; powered exponential model; positive-definite; random field; Whittle-Matérn class.

1 Introduction

Spatial data observed on the d-dimensional Euclidean space $\mathbb{R}^d$ are frequently modeled as the realizations of stationary and isotropic random fields. This approach requires the fitting of a correlation model. Specifically, a candidate function $\varphi : [0, \infty) \to \mathbb{R}$ is fitted to the observed correlations such that

$$\Big(\varphi(\|x_i - x_j\|)\Big)_{i,j=1}^{n} \tag{1}$$

is the correlation matrix for the random field restricted to any finite set $x_1, \ldots, x_n$ of points in $\mathbb{R}^d$. In other words, the correlation is supposed to depend on the Euclidean distance between x_i and x_j only. Isotropic models also form the building blocks of more sophisticated, nonisotropic or nonstationary models. The ingenious approach of Sampson and Guttorp (1992), e.g., deforms the geographic coordinate space into a new coordinate system where a stationary and isotropic correlation structure is modeled.

The availability of flexible, parameterized candidate models $\varphi : [0, \infty) \to \mathbb{R}$ has been of recent concern in spatial statistics as well as in various areas of application. We refer to Christakos (1984), Weber and Talkner (1993), Gaspari and Cohn (1999), Moreaux, Tscherning, and Sanso (1999), and the discussion on and rejoinder by Diggle, Tawn and Moyeed (1998). In the statistical literature, the powered exponential class,

$$\varphi_\alpha(t) = \exp(-t^\alpha), \qquad 0 < \alpha \leq 2, \tag{2}$$

and the Whittle-Matérn family,

$$\varphi_\mu(t) = \frac{2^{1-\mu}}{\Gamma(\mu)}\, t^\mu K_\mu(t), \qquad \mu > 0, \tag{3}$$

where K_μ is a modified Bessel function of index μ, have been especially popular. If μ is of the form $m + \frac{1}{2}$ where m is a nonnegative integer, then (3) reduces to the product of a polynomial of degree m in t and $\exp(-t)$, and we have $\varphi_{\frac{1}{2}}(t) = \exp(-t)$ and $\varphi_{\frac{3}{2}}(t) = (1+t)\exp(-t)$. The powered exponential family has been used by Hall and Wood (1993), Kent and Wood (1997), Diggle, Tawn and Moyeed (1998), and Davies and Hall (1999); and the Whittle-Matérn class (Whittle, 1954; Matérn, 1986, p. 18) has been employed in the works of Goff and Jordan (1988), Handcock and Wallis (1994), and Kent and Wood (1997), among others. These functions model strictly positive correlations. In contrast, the spherical model,

$$\varphi(t) = \begin{cases} 1 - \frac{3}{2}t + \frac{1}{2}t^3, & 0 \leq t \leq 1, \\ 0, & t \geq 1, \end{cases} \tag{4}$$

which is commonly used in environmental and geological sciences, has vanishing correlations beyond a cut-off distance. Indeed, in many applications compact support of the correlation function is highly desirable, for various ends.

Computationally efficient prediction and interpolation. Spatial prediction is the foremost goal in many studies. The kriging predictor $Z^*(x)$ for the realized value $Z(x)$ of the spatial variable at $x \in \mathbb{R}^d$ is the linear function of the variables $Z(x_1), \ldots, Z(x_n)$ at sampling locations $x_1, \ldots, x_n$ which minimizes the expected squared prediction error. Hence, the weights of the observations are derived from the estimated, isotropic correlation structure represented by φ. The formalism of dual kriging (Journel, 1989, pp. 14–15; Wackernagel, 1998, pp. 223–224) shows that kriging is equivalent to linear interpolation in terms of the radial basis functions $\varphi(\|x - x_1\|), \ldots, \varphi(\|x - x_n\|)$. Hence, if the dual approach is used compact support of φ reduces the computational burden of kriging. Compact support also allows for the use of computationally efficient sparse matrix techniques (Sansò and Schuh 1987, Barry and Pace 1997).

Fast and exact simulation. Spatial statistical inference and methodological studies frequently rely on simulated realizations of stationary random fields. For Gaussian simulations, the method of choice is clearly the circulant embedding approach of Wood and Chan (1994) and Dietrich and Newsam (1997). The technique is both fast and exact, but fails for certain correlation structures. It always works on sufficiently large simulation domains if φ has compact support (Dietrich and Newsam, 1997, pp. 1092 and 1098).

Appeal among practicioners. For many data sets, observed correlations essentially vanish beyond a certain cut-off distance. In meteorology, e.g., it is widely accepted that geopotential-height forecast error correlations should be set to zero beyond distances of a few thousand km in the troposphere (Gaspari and Cohn, 1999, pp. 750–751, and references therein). In the geological and environmental sciences, the popularity of the spherical model reflects similar beliefs and points at the intuitive appeal of compact support.

Yet compactly supported correlation functions have played very minor roles in the statistical literature. This seems to stem from a lack of flexible, compactly supported models other than the spherical function (4). The behavior of the correlation function at the origin largely reflects the smoothness of the random function model (Adler, 1981), and the interpolated surface formed by the kriging predictor inherits its smoothness from φ too. Thinking of φ as a function on $\mathbb{R}$ by setting $\varphi(-t) = \varphi(t)$, the associated stationary and isotropic random field is k times mean-square differentiable if and only if $\varphi^{(2k)}(0)$ exists. Furthermore, if $\varphi(t)$ is of the form

$$\varphi(t) = 1 - c|t|^{\alpha} + o(|t|^{\alpha}) \quad \text{as} \quad t \to 0 \tag{5}$$

for some $\alpha \in (0, 2]$ and $c > 0$, then with probability one the realizations of the associated Gaussian random field in $\mathbb{R}^d$ have fractal dimension $d + 1 - \frac{\alpha}{2}$. The value of α lies always between 0 and 2, and it is 2 if the random field is differentiable. The powered exponential family and the Whittle-Matérn class, for which $\alpha = 2\min(\mu, 1)$, permit the full range of allowable values for the fractal dimension. Moreover, the Whittle-Matérn model φ_μ has $2k$ derivatives at the origin if and only if $\mu > k$. In other words, these models parameterize the smoothness of the random field model. In contrast, compactly supported correlation functions which parameterize smoothness have not been available. The intention of the present article is to introduce this type of correlation model by straightforward constructions. We aim at applications in two different situations – studies in statistical methodology which rely on conveniently parameterized correlation models and exact simulation, and large-scale spatial prediction problems where compact support reduces the computational burden – but focus on mathematical aspects, and collate and extend existing literature in various scientific disciplines.

Specifically, Section 2 presents compactly supported correlation functions with an arbitrary but fixed number of derivatives at the origin. These functions are of simple analytical form and may be chosen as nonnegative and nonincreasing polynomials on their support. The construction is due to Wendland (1995) and has been adapted

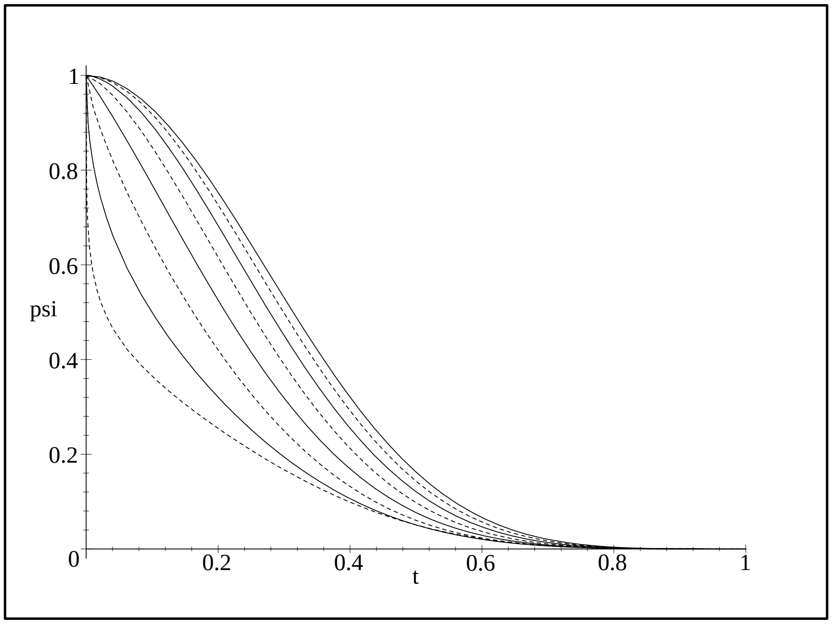

Figure 1: The product correlation function (6) for $\alpha = 2$, $\alpha = \frac{7}{4}$, …, $\alpha = \frac{1}{4}$, from top to bottom. The scale parameters are $L_\alpha = (2/\alpha)^{1/\alpha}$ and $L = 1$, and the model is permissible in three or less dimensions.

to statistical applications by Gneiting (1999c). Ramifications that allow for negative correlations are considered, too. In Section 3, we construct compactly supported correlation functions which parameterize smoothness for the random field model. Our suggestions include direct, compactly supported approximations to the powered exponential and Whittle-Matérn class as well as product correlation functions. Figure 1, e.g., illustrates the members of the family

$$\psi_\alpha(t) = \varphi_\alpha\left(\frac{t}{L_\alpha}\right) \varphi\left(\frac{t}{L}\right), \qquad 0 < \alpha \leq 2, \tag{6}$$

where φ_α is the exponential model (2), φ is Kanter's compactly supported function (22), and L and L_α are scale parameters. This family is compactly supported, parameterizes fractal dimension, and is permissible in three or less dimensions. The upper limit function ψ_2 is two and only two times differentiable at the origin and corresponds to a random field with one mean-square derivative, thereby avoiding the abrupt change from not differentiable to infinitely differentiable random functions at the upper limit of the powered exponential family.

Through the article, the argument t of the correlation function should be thought of as t/L with a scale parameter or cut-off value $L > 0$, which might be distinct for each factor in a product model of the type (6). Furthermore, unlike the spherical model, the vast majority of the models proposed here are twice differentiable with respect to the parameters, and thereby satisfy the conditions in the landmark paper of Mardia and Marshall (1984) on maximum likelihood estimation for spatial processes.

2 Construction of compactly supported correlation functions

2.1 Mathematical background

This section introduces notation and provides a condensed survey of background material. We denote by Φ_d the class of the continuous functions $\varphi : [0,\infty) \to \mathbb{R}$ which represent the correlation function of a stationary and isotropic random field on $\mathbb{R}^d$. In other words, the continuous function $\varphi : [0,\infty) \to \mathbb{R}$ belongs to Φ_d if and only if $\varphi(0) = 1$ and the matrix (1) is nonnegative definite for every finite system of points $x_1, \ldots, x_n$ in $\mathbb{R}^d$. Then $\Phi_{d+1} \subseteq \Phi_d$ for all d, and the elements of Φ_d are of the form

$$\varphi(t) = \int_{[0,\infty)} \Omega_d(tr)\, dF(r), \tag{7}$$

where F is a probability measure on $[0,\infty)$, and where

$$\Omega_d(t) = \Gamma(d/2) \left(\frac{2}{t}\right)^{(d-2)/2} J_{(d-2)/2}(t) \tag{8}$$

with $J_{(d-2)/2}$ a Bessel function of the first kind of order $(d-2)/2$. See, e.g., Matérn (1986, Section 2.3) and Gneiting and Sasvári (1999). The members of the powered exponential and Whittle-Matérn class belong to Φ_d for all d, and the spherical model (4) belongs to Φ_3 but not to Φ_4 (Gneiting, 1999a). Finally, it may be interesting to recall from Eq. (36) of Gneiting (1998) that the d-variate spectral density associated with an element φ of Φ_d is unimodal if and only if φ belongs to Φ_{d+2}, too.

The subsequent constructions rely on transformations between the classes Φ_d which preserve compact support. Specifically, we consider the descente,

$$I\varphi(t) = \int_t^\infty u\varphi(u)\, du \Big/ \int_0^\infty u\varphi(u)\, du, \qquad t \geq 0, \tag{9}$$

and the montée,

$$D\varphi(t) = \begin{cases} 1, & t = 0. \\ \varphi'(t)/(t\varphi''(0)), & t > 0. \end{cases}$$

The second derivative $\varphi''(0)$ refers to the symmetrically continued function; and under mild regularity conditions, I and D are inverse operators.

Theorem 1 (la descente) *Let φ be an element of Φ_d, $d \geq 3$. If $u\varphi(u)$ is integrable over $[0,\infty)$, then $I\varphi$ is an element of Φ_{d-2}.*

Proof. Suppose φ satisfies the conditions of the theorem, and define $\varphi_n(t) = \varphi(t)\exp(-t^2/n)$ for $n = 1, 2, \ldots$ By Lemma 2.1(2) of Wendland (1995), $I\varphi_n \in \Phi_{d-2}$ for all n; hence, $I\varphi = \lim_{n\to\infty} I\varphi_n$ belongs to Φ_{d-2}, too. □

Theorem 2 (le montée) *Let φ be an element of Φ_d. If $\varphi''(0)$ exists, then $D\varphi$ is an element of Φ_{d+2}.*

Proof. Suppose φ is an element of Φ_d for which $\varphi''(0)$ exists. By Lemma 3 of Gneiting (1999b), $\varphi''(0) = -d^{-1}\int_{[0,\infty)} r^2\, dF(r)$, where F is the probability measure in the canonical representation (7). In particular, F has a finite second moment. From standard properties of Bessel functions, $\Omega_d'(t) = -d^{-1}\, t\, \Omega_{d+2}(t)$, and the derivative is bounded. Thus, we may differentiate under the integral sign in the representation (7), so that

$$D\varphi(t) = \int_{[0,\infty)} \Omega_{d+2}(rt)\, r^2\, dF(r) \Big/ \int_{[0,\infty)} r^2\, dF(r)\ ;$$

this function is clearly an element of Φ_{d+2}. □

The theorems seem slightly stronger than previous results. Yet similar findings have been reported by various authors at various times and in various scientific disciplines. Matheron (1965, Chapter I) coined the terms descente and montée, which originate from an appealing physical interpretation in a mining context (see also Chilès and Delfiner, 1999, pp. 72–73). His results are very beautiful and will be rephrased here in terms of classes Φ_d, defined by Eqs. (7) and (8), with possibly non-integral index d. It can then be shown that $\Phi_{d'} \subseteq \Phi_d$ if $d' \geq d \geq 1$; and under suitable regularity conditions the operator

$$I^\kappa\varphi(t) = \int_t^\infty u\Big(u^2 - t^2\Big)^{\kappa-1}\varphi(u)\, du \Big/ \int_0^\infty u^{2\kappa-1}\varphi(u)\, du\ , \qquad t \geq 0, \qquad (10)$$

maps an element $\varphi \in \Phi_d$ to an element $I^\kappa\varphi \in \Phi_{d-2\kappa}$. Matheron provides a wealth of examples for the fractional descente (10) and its inverse, the fractional montée. His clavier de Bessel de 2^e espèce comprises the Whittle-Matérn class: if φ_μ is the Whittle-Matérn model (3) and $\kappa > 0$, then $I^\kappa\varphi_\mu = \varphi_{\mu+\kappa}$. Wu's (1995) recent construction corresponds to Matheron's (1965) clavier sphérique and includes the spherical function (4) and the cubic model (Wackernagel, 1998, p. 245, Chilès and Delfiner, 1999, p. 84) as special cases.

2.2 Wendland's construction

Here, we review Wendland's (1995) construction as adapted by Gneiting (1999c). The starting point is Golubov's (1981) result that the truncated power function

$$\varphi_{\nu,0}(t) = (1-t)_+^\nu = \begin{cases} (1-t)^\nu, & 0 \le t \le 1, \\ 0, & t \ge 1, \end{cases} \tag{11}$$

is an element of Φ_d if and only if $\nu \ge \frac{d+1}{2}$. Wendland (1995) then defined

$$\varphi_{\nu,k}(t) = I^k \varphi_{\nu,0}(t), \quad k = 0, 1, 2, \ldots, \tag{12}$$

by repeated application of the descente (9) to the truncated power function (11). By Theorems 1 and 2, $\varphi_{\nu,k}$ is a member of Φ_d if and only if $\nu \ge \frac{d+1}{2} + k$. Furthermore, $\varphi_{\nu,k}$ is $2k$ times differentiable at zero, positive and strictly decreasing on its support, and of the form $\varphi_{\nu,k}(t) = p_k(t)\,(1-t)_+^{\nu+k}$ with p_k a polynomial of degree k with coefficients in ν. Wendland focused on the case when ν is an integer, so that $\varphi_{\nu,k}$ is a polynomial on its support. He showed that the polynomial is of minimal degree for a given order of differentiability. Gneiting (1999c) pointed to the use of the parameter ν in covariance estimation and gave explicit formulas for the resulting parameterized families. Specifically,

$$\varphi_{\nu,1}(t) = \Big(1 + (\nu+1)\,t\Big)\,(1-t)_+^{\nu+1} \tag{13}$$

is an element of Φ_d if and only if $\nu \ge \frac{d+3}{2}$, and

$$\varphi_{\nu,2}(t) = \left(1 + (\nu+2)\,t + \frac{1}{3}\Big((\nu+2)^2 - 1\Big)\,t^2\right)(1-t)_+^{\nu+2} \tag{14}$$

belongs to Φ_d if and only if $\nu \ge \frac{d+5}{2}$. These functions have 2 and 4 derivatives at the origin, respectively. See Gneiting (1999c, Section 4) for illustrations and further discussion. Finally, one might generalize the construction and define $\varphi_{\nu,\kappa} = I^\kappa \varphi_{\nu,0}$ for $\kappa > 0$ by means of the fractional descente (10). We return to this approach in Section 3.1.

2.3 Hole effect models

The previous construction provides compactly supported, smooth correlation functions that decay monotonically to zero. In this section, we point at ramifications to model negative correlations, the so-called hole effect of geostatistics. Note that the representation (7) imposes a lower limit on the negative values attainable by isotropic correlation functions (Matérn, 1986, p. 16).

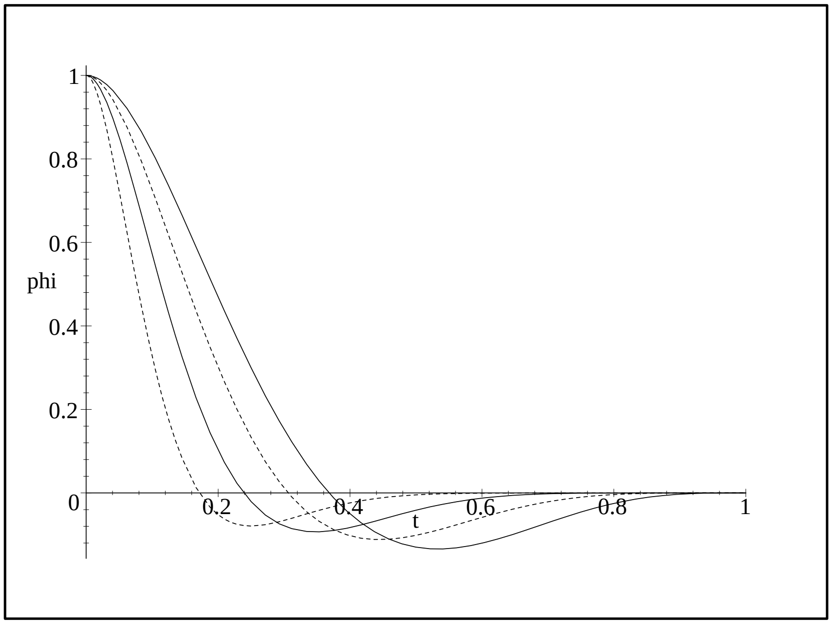

Figure 2: The hole effect model of Eq. (15) for $d = 2$ and $\nu = \frac{7}{2}$, $\nu = 5$, $\nu = \frac{15}{2}$, and $\nu = 12$, from top to bottom for small t.

The turning bands operator transforms a function $\varphi_d \in \Phi_d$, $d \geq 3$, into an element $\varphi_{d-2}(t) = \varphi_d(t) + (d-2)^{-1}\, t\, \varphi_d'(t)$ of Φ_{d-2}; and $\varphi_d \in \Phi_d$ if and only if $\varphi_{d-2} \in \Phi_{d-2}$ (Matheron, 1973). The operation preserves both compact support and the local behavior of the correlation function at the origin; furthermore, if φ_d is nonnegative then φ_{d-2} will often attain negative values. Applying the turning bands operator to the family in Eq. (13), e.g., results in

$$\tau_\nu(t) = \left(1 + \nu\, t - \frac{1}{d}\,(\nu+1)\,(\nu+2+d)\,t^2\right)(1-t)_+^\nu, \tag{15}$$

which is an element of Φ_d if and only if $\nu \geq \frac{d+5}{2}$. Figure 2 illustrates the members of this family for $d = 2$ and selected values of $\nu \geq \frac{7}{2}$.

As an alternative, Gaspari and Cohn (1999) construct compactly supported elements of Φ_3 which model negative correlations by direct convolution in $\mathbb{R}^3$. See Figures 7 and 8 of their work. Finally, a third approach is to consider product correlation functions: if $\psi(t) = \varphi(t)\,\Omega_d(t/L)$, $t \geq 0$, with a compactly supported, nonnegative function $\varphi \in \Phi_d$, Ω_d given by (8), and L a suitable length scale, then $\psi \in \Phi_d$ will display the hole effect. We refer to Section 3.3 for a discussion on the choice of φ.

3 Parameterizing smoothness

3.1 Compactly supported, exponential and Whittle-Matérn like functions

In this section, we seek compactly supported families within Φ_d that approximate the powered exponential class (2) and the Whittle-Matérn family (3) and provide a continuous parameterization of smoothness.

A natural candidate to approximate the powered exponential class is the truncated power family,

$$\varphi_{\alpha,\nu}(t) = (1 - t^\alpha)_+^\nu, \qquad 0 < \alpha < 2, \quad \nu \geq \nu_d(\alpha). \tag{16}$$

Clearly, $\varphi_{\alpha,\nu}(t/\nu^{1/\alpha}) \to \exp(-t^\alpha)$ as $\nu \to \infty$, uniformly in $t \geq 0$. The key question then is whether $\varphi_{\alpha,\nu}$ belongs to the class Φ_d of permissible correlation models. Golubov (1981) showed that for each positive integer d there exists a nondecreasing function $\nu_d(\alpha)$, $0 < \alpha < 2$, such that $\varphi_{\alpha,\nu} \in \Phi_d$ if and only if $\nu \geq \nu_d(\alpha)$. Furthermore, $\nu_d(\alpha) \geq \alpha + \frac{d-1}{2}$ with equality if $\alpha = 1$, and $\lim_{\alpha\to 2} \nu_d(\alpha) = \infty$. Finite upper bounds on $\nu_d(\alpha)$ have recently become available; e.g., $\nu_1(\frac{5}{3}) \leq 3$, $\nu_1(1.9550) \leq 10$, and $\nu_3(1.8095) \leq 6$ (Gneiting, 2000a, 2000b). Yet for any given ν the permissible functions of the form (16) cover only part of the allowable range $0 < \alpha \leq 2$ for the local behavior (5) at the origin, and therefore for the fractal dimension.

To approximate the Whittle-Matérn family (3), let us return to the construction in Section 2.2 and apply the fractional descente (10) to the truncated power model (11), so that

$$\varphi_{\nu,\kappa}(t) = I^\kappa \varphi_{\nu,0}(t) = c^{-1} \int_t^1 u\left(u^2 - t^2\right)^{\kappa-1} (1-u)_+^\nu \, du, \qquad \kappa > 0. \tag{17}$$

Here c is the Beta integral $B(2\kappa, \nu + 1)$; and $\varphi_{\nu,\kappa}$ belongs to Φ_d if and only if $\nu \geq \frac{d+1}{2} + \kappa$. Moreover, since $I^\kappa \varphi_\mu = \varphi_{\mu+\kappa}$ for the Whittle-Matérn model (3), and $\lim_{\nu\to\infty} \varphi_{\nu,0}(t/\nu) = \exp(-t) = \varphi_{\frac{1}{2}}(t)$, uniformly in $t \geq 0$, it is immediate that $\lim_{\nu\to\infty} \varphi_{\nu,\kappa}(t/\nu) = \varphi_{\frac{1}{2}+\kappa}(t)$, uniformly in $t \geq 0$. Finally, $\varphi_{\nu,\kappa}$ and the Whittle-Matérn model $\varphi_{\frac{1}{2}+\kappa}$ have the same type of behavior at the origin.

Unfortunately, closed form solutions of the fractional integral in (17) are not available unless κ is an integer, or ν is an integer and $\kappa = \frac{1}{2}, \frac{3}{2}, \ldots$ If $\kappa < 1$, a straightforward change of coordinates removes the singularity of the integrand, so that numerical integration is feasible but cumbersome. In this light, we turn our attention to compactly supported product correlation functions, an approach proposed by Gaspari and Cohn (1999, Section 4d).

3.2 Product correlation functions

Suppose $\{\varphi_\alpha : 0 < \alpha \le 2\} \subseteq \Phi_d$ is a parameterized family of correlation models with a local behavior of the type (5) at the origin, i.e., $\varphi_\alpha(t) = 1 - c|t|^\alpha + o(|t|^\alpha)$ as $t \to 0$, for a constant c which may depend on α. Let φ be a compactly supported element of Φ_d which is twice differentiable at the origin. Since Φ_d is closed under multiplication, the product function

$$\psi_\alpha(t) = \varphi_\alpha\left(\frac{t}{L_\alpha}\right)\varphi\left(\frac{t}{L}\right), \qquad 0 < \alpha \le 2, \tag{18}$$

where L_α and L are scale parameters, belongs to Φ_d too. Furthermore, ψ_α has compact support and behaves like $1 - c|t|^\alpha + o(|t|^\alpha)$ as $t \to 0$. In other words, $\{\psi_\alpha\}$ parameterizes fractal dimension in the same manner as the original family $\{\varphi_\alpha\}$ does.

Similarly, if φ_μ is the Whittle-Matérn model (3) and φ is a compactly supported element of Φ_d for which $\varphi^{(2k)}(0)$ exists, then the product function

$$\psi_\mu(t) = \varphi_\mu\left(\frac{t}{L_\mu}\right)\varphi\left(\frac{t}{L}\right), \qquad \mu > 0, \tag{19}$$

is a compactly supported element of Φ_d; and if φ_μ has $2l \le 2k$ derivatives at the origin then so has ψ_μ. Thus, $\{\psi_\mu\}$ provides a continuous parameterization of random functions whose realizations are l times differentiable, $l = 0, 1, \ldots, k$.

Ideally, one would choose the compactly supported factor φ as an infinitely differentiable function, so that the product model (19) permits the full range of smoothness allowed by the Whittle-Matérn family. Clearly, compactly supported members of Φ_d with infinitely many derivatives exist, such as the suitably normalized self-convolution of an infinitely differentiable, radial function supported on a ball in $\mathbb{R}^d$. However, the author does not know of a closed-form representation for a self-convolution of this type. From an applied point of view, an upper bound on the smoothness of the random field does not appear to pose major problems. In simulation studies, an arbitrarily high upper bound is certainly not of concern. In covariance fitting, it is frequently reasonable to assume a priori that high values of the smoothness parameter are less likely, as in Handcock and Wallis (1994, p. 373) and Handcock (1998). Furthermore, one might well use the Whittle-Matérn model (3) in an initial stage of the model fitting and shift to compactly supported models at a later stage only, which retains the benefits of either approach. Clearly, there is much scope for methodological research in the area, and experiments with real and simulated data sets, as in Dee and others (1999), are desirable.

3.3 Examples of product correlation functions

In this final section, we discuss the specific choice of the two factors in a product of the form (18) or (19).

The Whittle-Matérn class (3) is indeed the only candidate for an analytically tractable, continuously parameterized family $\{\varphi_\mu : \mu > 0\}$ of correlation functions, which admit the entire range of $k = 0, 1, 2, \ldots$ times differentiable random fields. For an embedding into a larger class of parameterized correlation functions see Section 3 of Gneiting (1999c). The powered exponential class is the standard model for a parameterization $\{\varphi_\alpha : 0 < \alpha \leq 2\}$ of fractal dimension. Yet there are other options, such as the Cauchy family

$$\varphi_{\alpha,\beta}(t) = (1 + t^\alpha)^{-\beta/\alpha}, \qquad 0 < \alpha \leq 2, \quad \beta > 0. \tag{20}$$

These functions belong to Φ_d for all d, because $\varphi_{\alpha,\beta}$ has a scale mixture representation in terms of the upper limit function $\varphi_2(t) = \exp(-t^2)$ of the powered exponential class (Gneiting, 1997, Example 4). Again, the fractal dimension of the realizations is $d + 1 - \frac{\alpha}{2}$, whereas β is a long-memory parameter. In fact, the Cauchy family seems to provide the power-law correlation function with non-integral index β, for which Whittle (1962, p. 314) had called. Beran (1994, Sections 1.3.5 and 1.5.3) gives a more detailed discussion of long-memory dependence in the random field context.

How should the compactly supported and smooth second factor φ be chosen? If the product correlation function (18) or (19) is supposed to approximate $\{\varphi_\alpha\}$ or $\{\varphi_\mu\}$, respectively, it is natural to minimize the curvature of φ at zero (cf. Gaspari and Cohn, 1999, Section 4d). To fix the idea, denote by Φ_d^0 the class of all the functions $\varphi \in \Phi_d$ such that $\varphi(t) = 0$ for $t \geq 1$ and $\varphi''(0)$ exists. We seek to minimize $|\varphi''(0)|$ within the classes Φ_d^0, $d \geq 1$. The problem has been solved if $d = 1$ and if $d = 3$.

Theorem 3 *If $\varphi \in \Phi_1^0$, then $|\varphi''(0)| \geq \pi^2$ with equality if and only if*

$$\varphi(t) = (1 - t)\cos(\pi t) + \frac{1}{\pi}\sin(\pi t), \qquad 0 \leq t \leq 1. \tag{21}$$

This result is due to Bohman (1960), and the compactly supported function (21) is illustrated in Figure 3.

Theorem 4 *If $\varphi \in \Phi_3^0$, then $|\varphi''(0)| \geq \frac{4}{3}\pi^2$ with equality if and only if*

$$\varphi(t) = (1 - t)\,\frac{\sin(2\pi t)}{2\pi t} + \frac{1}{\pi}\,\frac{1 - \cos(2\pi t)}{2\pi t}, \qquad 0 \leq t \leq 1. \tag{22}$$

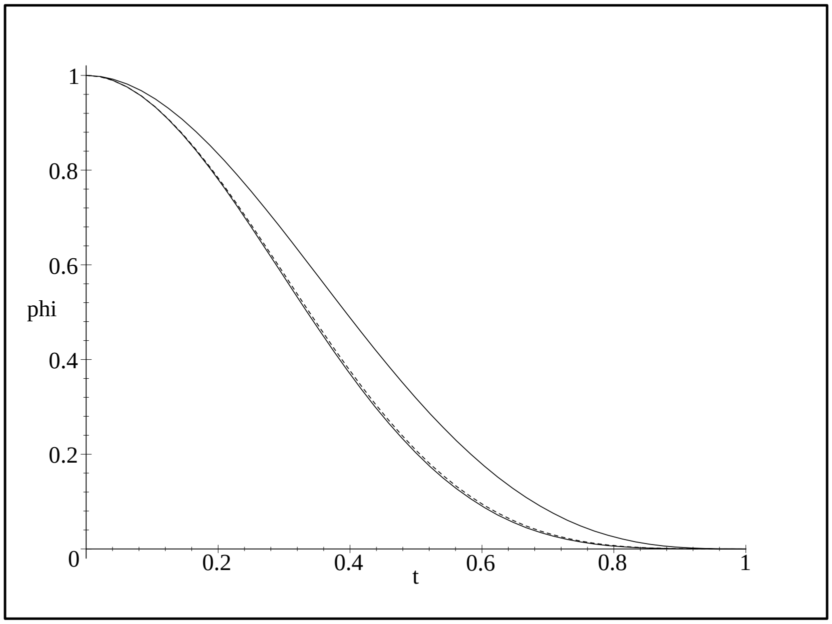

Figure 3: The extremal functions (21) and (22) of Theorem 3 and 4, solid lines, from top to bottom; and the function (23) of Gaspari and Cohn, broken line.

To prove Theorem 4, recall from Eq. (36) of Gneiting (1998) that Φ_3^0 can be identified with the class of functions $\varphi \in \Phi_1^0$ for which the associated spectral density is unimodal. Kanter (1997) solves the extremal problem for the latter class, and his solution is given here.

For $d = 2$ and $d \geq 4$, the problem is open. However, a very interesting construction is due to Gaspari and Cohn (1999, Section 4d). They propose to use a product correlation function of the type $\psi(t) = (1 + (t/L_1)^2)^{-1}\varphi(t/L_2)$ in atmospheric data analysis systems, where L_1 and L_2 are scale parameters, and

$$\varphi(t) = \begin{cases} 1 - \frac{20}{3}t^2 + 5t^3 + 8t^4 - 8t^5, & 0 \leq t \leq \frac{1}{2}, \\ \frac{1}{3}\,t^{-1}\left(8t^2 + 8t - 1\right)\left(1 - t\right)^4, & \frac{1}{2} \leq t \leq 1, \end{cases} \tag{23}$$

is an element of Φ_3^0 for which $|\varphi''(0)| = \frac{40}{3}$ is surprisingly close to the lower bound of Theorem 4. Furthermore, Figure 3 illustrates that (23) attains larger values than the extremal function (22) on the joint support, except when $t < 0.0617\ldots$

These results clearly suggest the use of the functions (21), (22), and (23), respectively, as the compactly supported factor in a product correlation function of the type (18). For products of the type (19), smoother factors with $2k \geq 4$ derivatives at the origin may be needed. Then Wendland's function (12) with the minimal value of ν for

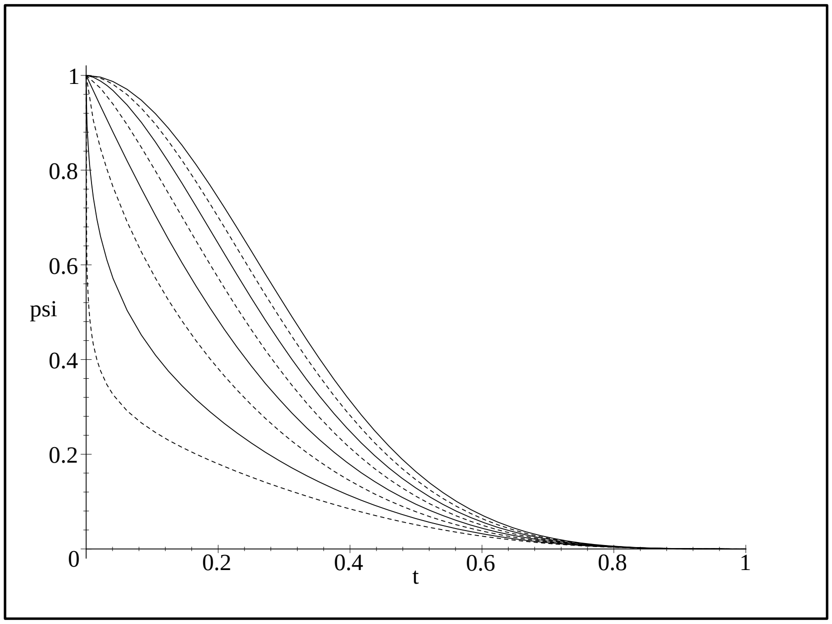

Figure 4: The product correlation function (24) for $\alpha = 2$, $\alpha = \frac{7}{4}$, $\ldots$, $\alpha = \frac{1}{4}$, from top to bottom. These functions are permissible in one dimension.

the given dimension d and smoothness k might be chosen. Alternatively, choosing the minimal value of ν for dimension $d+2$ ensures that the compactly supported factor has a unimodal spectral density. Clearly, this carries over to the product correlation function if the spectral density of the second factor is unimodal, too.

We close the paper with a discussion of some specific product correlation functions. The model (6) that we saw in Section 1 is the product of the powered exponential (2) and Kanter's function (22). Again, we stress that the compactly supported product model avoids the abrupt change from not differentiable to infinitely differentiable random functions at the upper limit, which is frequently considered a drawback of the powered exponential class. Figure 4 illustrates the product model

$$\psi_\alpha(t) = (1 + t^\alpha)^{-3} \left((1-t)\cos(\pi t) + \frac{1}{\pi}\sin(\pi t) \right), \qquad 0 < \alpha \le 2, \tag{24}$$

for $t \le 1$, and 0 otherwise, which is also of the form (18). Here $\{\varphi_\alpha\}$ is the Cauchy family (20) with $\beta = 3\alpha$, φ is Bohman's function (21), and $L_\alpha = L = 1$. Clearly, this correlation model is valid if $d = 1$ only. Finally, Figure 5 displays the family

$$\psi_\mu(t) = \varphi_\mu(t) \left(1 + \frac{11}{2}t + \frac{117}{12}t^2 \right) (1-t)_+^{11/2}, \qquad \mu > 0, \tag{25}$$

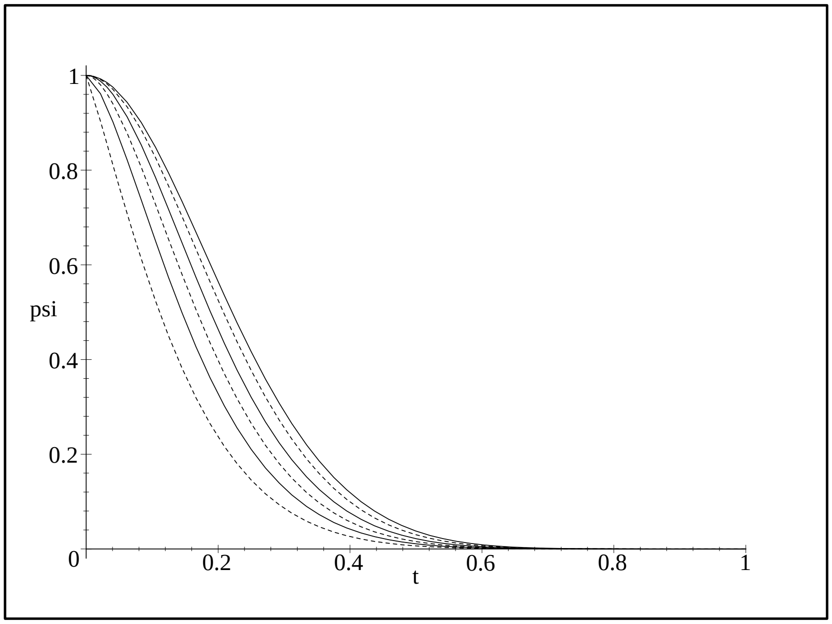

Figure 5: The product correlation function (25) for $\mu = \frac{5}{2}$, $\mu = 2$, $\mu = \frac{3}{2}$, $\mu = \frac{11}{10}$, $\mu = \frac{4}{5}$, and $\mu = \frac{6}{11}$, from top to bottom. This model is permissible in one- or two-dimensional space.

where φ_μ is the Whittle-Matérn model (3). This is of the type (19) with $L_\mu = L = 1$, and $\varphi = \varphi_{\frac{7}{2},2}$ an instance of the family (14) whose elements are four times differentiable at zero. The product model is permissible in one or two dimensions and has up to four derivatives at the origin, too.

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References

Adler, R. J. (1981) *The Geometry of Random Fields.* John Wiley, Chichester.

Barry, R. P and Pace, R. K. (1997). Kriging with large data sets using sparse matrix techniques. *Commun. Statist. Simul. Comput.* **26**, 619–629.

Beran, J. (1994) *Statistics for Long-memory Processes.* Chapman and Hall, London.

Bohman, H. (1960) Approximate Fourier analysis of distribution functions. *Ark. Mat.* **4**, 99–157.

Chilès, J.-P. and Delfiner, P. (1999) *Geostatistics. Modeling Spatial Uncertainty.* Wiley, New York.

Christakos, G. (1984) On the problem of permissible covariance and variogram models. *Water Resour. Res.* **20**, 251–265.

Davies, S. and Hall, P. (1999) Fractal analysis of surface roughness by using spatial data (with discussion). *J. R. Statist. Soc.* B **61**, 3–37.

Dee, D. P., Gaspari, G., Redder, Ch., Rukhovets, L. and da Silva, A. M. (1999) Maximum-likelihood estimation of forecast and observation error covariance parameters. Part II: applications. *Mon. Wea. Rev.* **127**, 1835–1849.

Dietrich, C. R. and Newsam, G. N. (1997) Fast and exact simulation of stationary Gaussian processes through circulant embedding of the covariance matrix. *SIAM J. Sci. Comput.* **18**, 1088–1107.

Diggle, P., Tawn, J. A. and Moyeed, R. A. (1998) Model-based geostatistics (with discussion). *Appl. Statist.* **47**, 299–350.

Gaspari, G. and Cohn, S. E. (1999) Construction of correlation functions in two and three dimensions. *Quart. J. R. Meteorol. Soc.* **125**, 723–757.

Gneiting, T. (1997) Normal scale mixtures and dual probability densities. *J. Statist. Comput. Simulation* **59**, 375-384.

Gneiting, T. (1998) On α-symmetric multivariate characteristic functions. *J. Multivar. Anal.* **64**, 131-147.

Gneiting, T. (1999a) Radial positive definite functions generated by Euclid's hat. *J. Multivar. Anal.* **69**, 88-119.

Gneiting, T. (1999b) On the derivatives of radial positive definite functions. *J. Math. Anal. Appl.* **236**, 86–93.

Gneiting, T. (1999c) Correlation functions for atmospheric data analysis. *Quart. J. R. Meteorol. Soc.* **125**, 2449-2464.

Gneiting, T. (2000a) Kuttner's problem and a Pólya type criterion for characteristic functions. *Proc. Am. Math. Soc.* **128**, 1721–1728.

Gneiting, T. (2000b) Criteria of Pólya type for radial positive definite functions. To appear in *Proc. Am. Math. Soc.*

Gneiting, T. and Sasvári, Z. (1999) The characterization problem for isotropic covariance functions. *Math. Geol.* **31**, 105–11.

Goff, J. A. and Jordan, T. H. (1988) Stochastic modeling of seafloor morphology: inversion of sea beam data for second-order statistics. *J. Geophys. Res.* **93**, 13589–13608.

Golubov, B. I. (1981) On Abel-Poisson type and Riesz means. *Anal. Math.* **7**, 161–184.

Hall, P. and Wood, A. (1993). On the performance of box-counting estimators of fractal dimension. *Biometrika* **80**, 246–252.

Handcock, M. S. (1998) Discussion on the paper by Diggle et al. *Appl. Statist.* **47**, 337–338.

Handcock, M. S. and Wallis, J. R. (1994) An approach to statistical spatial-temporal modeling of meteorological fields (with discussion). *J. Amer. Statist. Assoc.* **89**, 368–390.

Journel, A. G. (1989) *Fundamentals of Geostatistics in Five Lessons.* American Geophysical Union, Washington, D.C.

Kanter, M. (1997) Unimodal spectral windows. *Statist. Prob. Lett.* **34**, 403–411.

Kent, J. T. and Wood, A. T. A. (1997) Estimating the fractal dimension of a locally self-similar Gaussian process by using increments. *J. R. Statist. Soc.* B **59**, 679–699.

Mardia, K. V. and Marshall, R. J. (1984) Maximum likelihood estimation of models for residual covariance in spatial regression. *Biometrika* **71**, 135–146.

Matérn, B. (1986) *Spatial Variation.* 2nd ed., Lecture Notes in Statistics **36**. Springer, Berlin.

Matheron, G. (1965) *Les variables régionalisées et leur estimation*. Masson, Paris.

Matheron, G. (1973) The intrinsic random functions and their applications. *Adv. Appl. Prob.* **5**, 439–468.

Moreaux, G., Tscherning, C. C., and Sanso, F. (1999) *J. Geodesy* **73**, 555–567.

Sampson, P. D. and Guttorp, P. (1992) Nonparametric estimation of nonstationary spatial covariance structure. *J. Amer. Statist. Assoc.* **87**, 108–119.

Sansò, F. and Schuh, W.-D. (1987) Finite covariance functions. *Bull. Géodesique* **61**, 331–347.

Wackernagel, H. (1998) *Multivariate Geostatistics*, second ed. Springer, Berlin.

Weber, R. O. and Talkner, P. (1993) Some remarks on spatial correlation function models. *Mon. Wea. Rev.* **121**, 2611–2617, **127**, 576.

Wendland, H. (1995) Piecewise polynomial, positive definite and compactly supported radial functions of minimal degree. *Adv. Comput. Math.* **4**, 389–396.

Whittle, P. (1954) On stationary processes in the plane. *Biometrika* **41**, 434–449.

Whittle, P. (1962) Topographic correlation, power-law covariance functions, and diffusion. *Biometrika* **49**, 305–314.

Wood, A. T. A. and Chan, G. (1994) Simulation of stationary processes in $[0,1]^d$. *J. Comput. Graph. Statist.* **3**, 409–432.

Wu, Z. (1995) Compactly supported positive definite radial functions. *Adv. Comput. Math.* **4**, 283–292.