Haar function construction of Brownian bridge and Brownian motion

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Existence of Brownian motion and Brownian bridge as continuous processes on $C[0,1]$

The aim of this subsection to convince you that both Brownian motion and Brownian bridge exist as *continuous Gaussian processes* on $[0,1]$, and that we can then extend the definition of Brownian motion to $[0,\infty)$.

Definition 1. *Brownian motion* (or *standard Brownian motion*, or a Wiener process) $\mathbb{S}$ is a Gaussian process with continuous sample functions and:
(i) $\mathbb{S}(0)=0$;
(ii) $E(\mathbb{S}(t))=0, \quad 0\le t\le 1$;
(iii) $E\{\mathbb{S}(s)\mathbb{S}(t)\}=s\wedge t, \quad 0\le s,t\le 1$.

Definition 2. A *Brownian bridge process* $\mathbb{U}$ is a Gaussian process with continuous sample functions and:
(i) $\mathbb{U}(0)=\mathbb{U}(1)=0$;
(ii) $E(\mathbb{U}(t))=0, \quad 0\le t\le 1$;
(iii) $E\{\mathbb{U}(s)\mathbb{U}(t)\}=s\wedge t-st, \quad 0\le s,t\le 1$.

Theorem 1. Brownian motion $\mathbb{S}$ and Brownian bridge $\mathbb{U}$ exist.

Proof. We first construct a Brownian bridge process $\mathbb{U}$. Let

$$h_{00}(t)\equiv h(t)\equiv \begin{cases} t & 0\le t\le 1/2\,,\\ 1-t & 1/2\le t\le 1\,,\\ 0 & \text{elsewhere}\,. \end{cases} \tag{1}$$

For $n\ge 1$ let

$$h_{nj}(t)\equiv 2^{-n/2}h(2^n t-j), \qquad j=0,\ldots,2^n-1\,. \tag{2}$$

For example, $h_{10}(t)=2^{-1/2}h(2t)$, $h_{11}(t)=2^{-1/2}h(2t-1)$, while

$$h_{20}(t)=2^{-1}h(4t), \qquad h_{21}(t)=2^{-1}h(4t-1)\,,$$
$$h_{22}(t)=2^{-1}h(4t-2), \qquad h_{23}(t)=2^{-1}h(4t-3)\,.$$

Note that $|h_{nj}(t)|\le 2^{-n/2}2^{-1}$.

The functions $\{h_{nj}:\ j=0,\ldots,2^n-1,\ \ n\ge 0\}$ are called the *Schauder functions*; they are integrals of the orthonormal (with respect to Lebesgue measure on $[0,1]$) family of functions $\{g_{nj}: \; j=0,\ldots,2^n-1, \;\; n \geq 0\}$ called the *Haar functions* defined by

$$\begin{aligned} g_{00}(t) &\equiv g(t) \equiv 2 1_{[0,1/2]}(t) - 1, \\ g_{nj}(t) &\equiv 2^{n/2} g_{00}(2^n t - j)\,, \qquad j = 0,\ldots,2^n-1, \;\; n \geq 1\,. \end{aligned}$$

Thus

$$\int_0^1 g_{nj}^2(t)dt = 1, \qquad \int_0^1 g_{nj}(t) g_{n'j'}(t)dt = 0 \qquad \text{if} \qquad n \neq n', \text{ or } \;\; j \neq j'\,, \tag{3}$$

and

$$h_{nj}(t) = \int_0^t g_{nj}(s)ds, \qquad 0 \leq t \leq 1\,. \tag{4}$$

Furthermore, the family $\{g_{nj}\}_{j=0,\,n\geq 0}^{2^n-1} \cup \{g(\cdot/2)\}$ is complete: any $f \in L_2(0,1)$ has an expansion in terms of the g's. In fact the Haar basis is the simplest *wavelet basis* of $L_2(0,1)$, and is the starting point for further developments in the area of wavelets.

Now let $\{Z_{nj}\}_{j=0,\,n\geq 0}^{2^n-1}$ be independent identically distributed $N(0,1)$ random variables; if we wanted, we could construct all these random variables on the probability space $([0,1], \mathcal{B}_{[0,1]}, \lambda)$. Define

$$\begin{aligned} V_n(t,\omega) &= \sum_{j=0}^{2^n-1} Z_{nj}(\omega) h_{nj}(t)\,, \\ U_m(t,\omega) &= \sum_{n=0}^{m} V_n(t,\omega)\,. \end{aligned}$$

For $m > k$

$$|U_m(t,\omega) - U_k(t,\omega)| = |\sum_{n=k+1}^{m} V_n(t,\omega)| \leq \sum_{n=k+1}^{m} |V_n(t,\omega)| \tag{5}$$

where

$$|V_n(t,\omega)| \leq \sum_{j=0}^{2^n-1} |Z_{nj}(\omega)||h_{nj}(t)| \leq 2^{-(n/2+1)} \max_{0\leq j\leq 2^n-1} |Z_{nj}(\omega)| \tag{6}$$

since the h_{nj}, $j = 0,\ldots,2^n-1$ are $\neq 0$ on disjoint t intervals.

Now $P(Z_{nj} > z) = 1 - \Phi(z) \le z^{-1}\phi(z)$ for $z > 0$ (by "Mill's ratio") so that

$$P(|Z_{nj}| \ge 2\sqrt{n}) = 2P((Z_{nj} \ge 2\sqrt{n}) \le \frac{2}{\sqrt{2\pi}}(2\sqrt{n})^{-1}e^{-2n}\,. \tag{7}$$

Hence

$$P\left(\max_{0\le j\le 2^n-1} |Z_{nj}| \ge 2\sqrt{n}\right) \le 2^n P(|Z_{00}| \ge 2\sqrt{n}) \le \frac{2^n}{\sqrt{2\pi}} n^{-1/2}e^{-2n}\,; \tag{8}$$

since this is a term of a convergent series, by the Borel-Cantelli lemma $\max_{0\le j\le 2^n-1}|Z_{nj}| \ge 2\sqrt{n}$ occurs infinitely often with probability zero; i.e. except on a null set, for all ω there is an $N = N(\omega)$ such that $\max_{0\le j\le 2^n-1}|X_{nj}(\omega)| < 2\sqrt{n}$ for all $n > N(\omega)$. Hence

$$\sup_{0\le t\le 1} |U_m(t) - U_k(t)| \le \sum_{n=k+1}^{m} 2^{-n/2}n^{1/2} \downarrow 0 \tag{9}$$

for all $k, m \ge N' \ge N(\omega)$. Thus $U_m(t,\omega)$ converges uniformly as $m \to \infty$ with probability one to the (necessarily continuous) function

$$\mathbb{U}(t,\omega) \equiv \sum_{n=0}^{\infty} V_n(t,\omega)\,. \tag{10}$$

Define $\mathbb{U} \equiv 0$ on the exceptional set. Then $\mathbb{U}$ is continuous for all ω.

Now $\{\mathbb{U}(t):\ 0 \le t \le 1\}$ is clearly a Gaussian process since it is the sum of Gaussian processes. We now show that $\mathbb{U}$ is in fact a Brownian bridge: by formal calculation (it remains only to justify the interchange of summation and expectation),

$$
\begin{aligned}
E\{\mathbb{U}(s)\mathbb{U}(t)\} &= E\left\{\sum_{n=0}^{\infty} V_n(s) \sum_{m=0}^{\infty} V_m(t)\right\} \\
&= \sum_{n=0}^{\infty} E\{V_n(s)V_n(t)\} \\
&= \sum_{n=0}^{\infty} E\left\{\sum_{j=0}^{2^n-1} Z_{nj} \int_0^s g_{nj}d\lambda \sum_{k=0}^{2^n-1} Z_{nk} \int_0^t g_{nk}d\lambda\right\} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \int_0^s g_{nj}d\lambda \int_0^t g_{nj}d\lambda \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} \int_0^1 1_{[0,s]}g_{nj}d\lambda \int_0^1 1_{[0,t]}g_{nj}d\lambda + st - st \\
&= \int_0^1 1_{[0,s]}(u)1_{[0,t]}(u)du - st \\
&= s \wedge t - st
\end{aligned}
$$

where the next to last equality follows from Parseval's identity. Thus $\mathbb{U}$ is Brownian bridge.

Now let Z be one additional $N(0,1)$ random variable independent of all the others used in the construction, and define

$$
\mathbb{S}(t) \equiv \mathbb{U}(t) + tZ = \sum_{n=0}^{\infty} V_n(t) + tZ\,. \tag{11}
$$

Then $\mathbb{S}$ is also Gaussian with 0 mean and

$$
\begin{aligned}
Cov[\mathbb{S}(s), \mathbb{S}(t)] &= Cov[\mathbb{U}(s) + sZ, \mathbb{U}(t) + tZ] \\
&= Cov[\mathbb{U}(s), \mathbb{U}(t)] + stVar(Z) \\
&= s \wedge t - st + st = s \wedge t\,.
\end{aligned}
$$

Thus $\mathbb{S}$ is Brownian motion. Since $\mathbb{U}$ has continuous sample paths, so does $\mathbb{S}$. □

The following figures illustrate the construction given in the theorem.

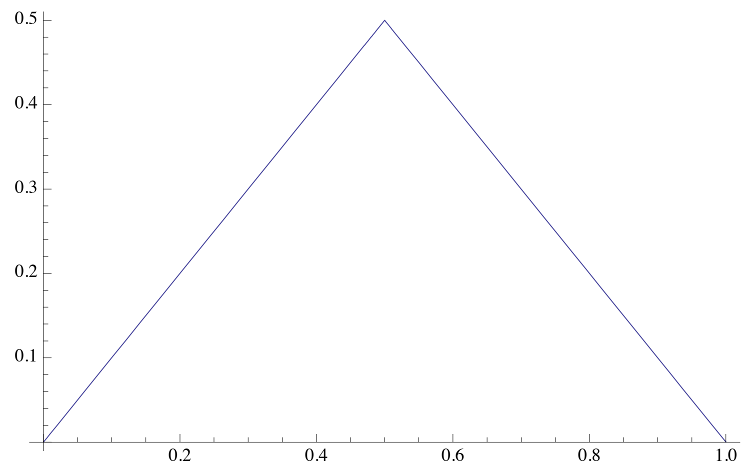

Figure 1: The Schauder function h_{00}.

Here is the Mathematica code that produced the plots for the case $m = 8$.

```
Needs["Histograms`"]
ndist = NormalDistribution[0, 1]
m = 8
Z = Table[RandomReal[ndist, 2^n], {n, 0, m}]
Histogram[Flatten[Z]]

h[t_] := t /; 0 <= t <= 1/2
h[t_] := 1 - t /; 1/2 < t <= 1
h[t_] := 0 /; t < 0 || t > 1
h1[t_, n_, j_] := 2^(-n/2)*h[2^n *t - j]
V0[t_] := Z[[1, 1]]*h[t]
V[t_, n_] := Sum[Z[[n + 1, j + 1]]*h1[t, n, j], {j, 0, 2^n - 1}]
U[t_, m_] := Sum[V[t, n], {n, 1, m}] + V0[t]
P1 = Plot[h[t], {t, 0, 1}]
P2 = Plot[h1[t, 1, 0], {t, 0, 1}]
P3 = Plot[h1[t, 1, 1], {t, 0, 1}]
Show[P2, P3]
P4 = Plot[h1[t, 2, 0], {t, 0, 1}, PlotRange -> {0, .5}]
P5 = Plot[h1[t, 2, 1], {t, 0, 1}, PlotRange -> {0, .5}]
P6 = Plot[h1[t, 2, 2], {t, 0, 1}, PlotRange -> {0, .5}]
```

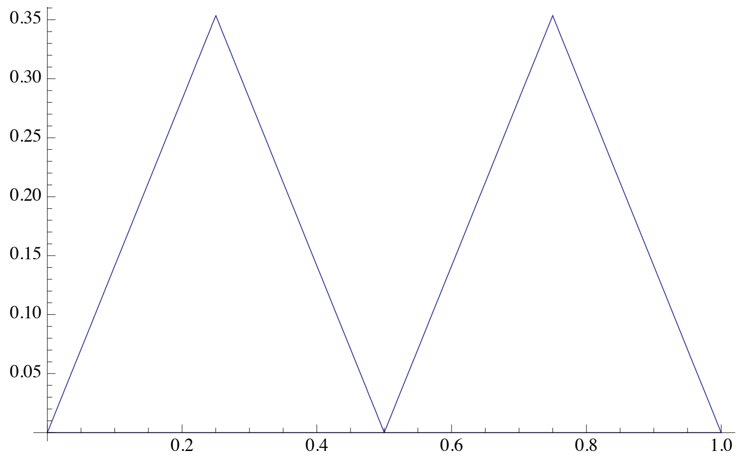

Figure 2: The Schauder functions h_{10} and h_{11}.

```
P7 = Plot[h1[t, 2, 3], {t, 0, 1}, PlotRange -> {0, .5}]
Show[P4, P5, P6, P7]
Plot[V0[t], {t, 0, 1}]
Plot[V[t, 1], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.0, 0.0, 1.00],
Plot[V[t, 2], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.1, 0.00, .90],
Plot[V[t, 3], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.2, 0.00, .80],
Plot[V[t, 4], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.3, 0.00, .70],
Plot[V[t, 5], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.4, 0.00, .60],
Plot[V[t, 6], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.5, 0.00, .50],
Plot[V[t, 7], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.6, 0.00, .40],
Plot[V[t, 8], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.7, 0.00, .30],

Plot[U[t, 8], {t, 0, 1}, PlotStyle -> {Thickness[1/200], RGBColor[0.0, 0.50, .50],
```

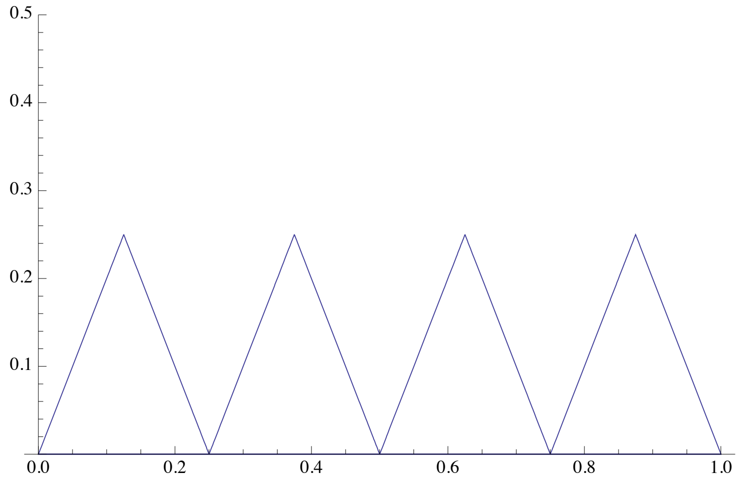

Figure 3: The Schauder functions h_{20}, h_{21}, h_{22}, h_{23}.

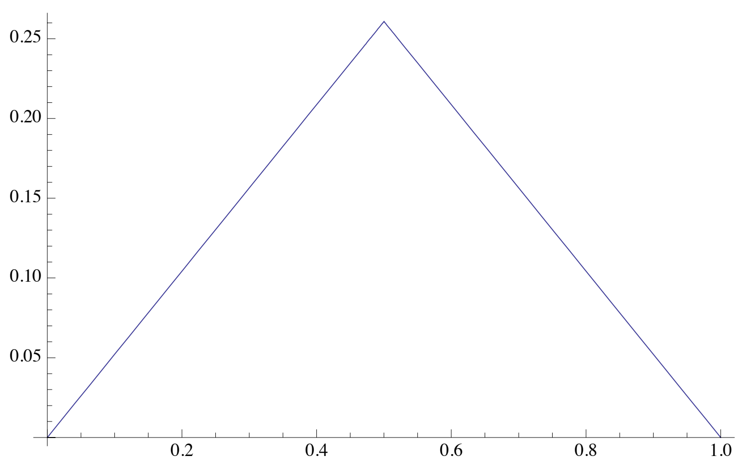

Figure 4: A sample path of the random function $V_0(t)$.

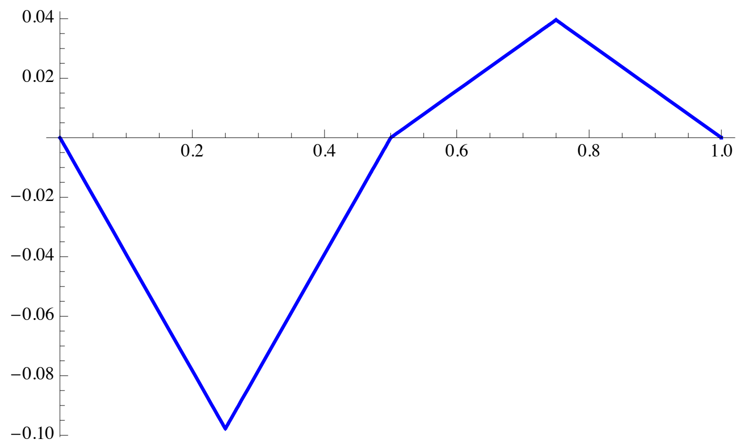

Figure 5: A sample path of the random function $V_1(t)$.

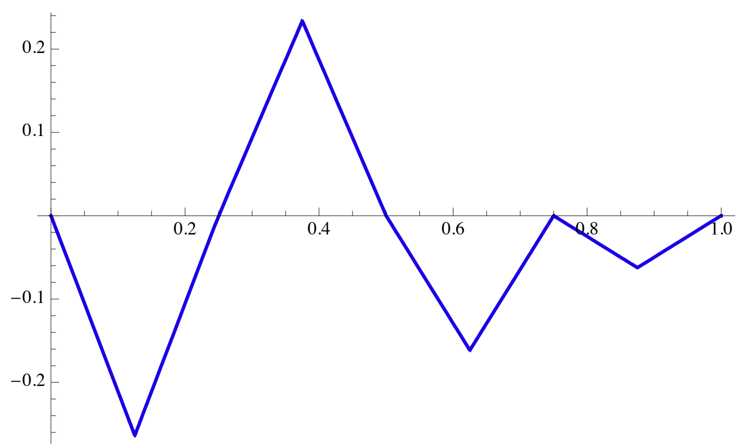

Figure 6: A sample path of the random function $V_2(t)$.

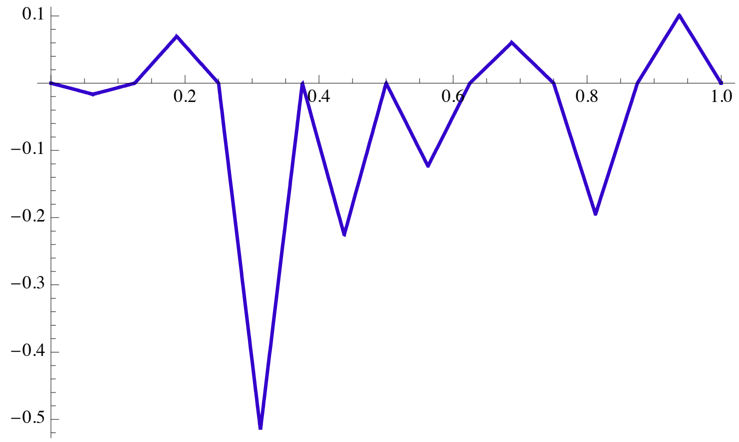

Figure 7: A sample path of the random function $V_3(t)$.

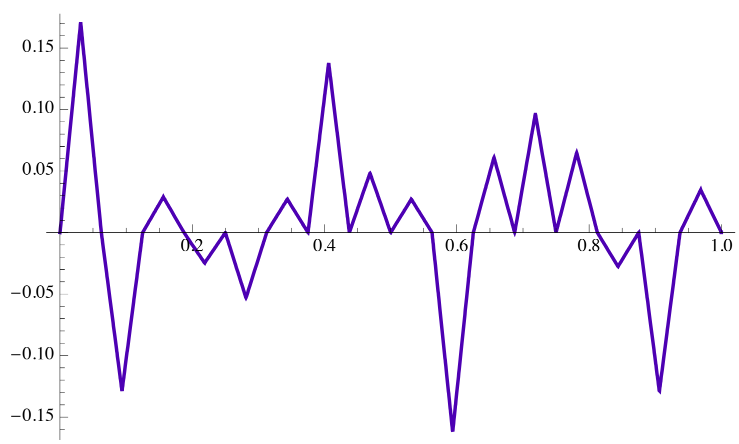

Figure 8: A sample path of the random function $V_4(t)$.

Figure 9: A sample path of the random function $V_5(t)$.

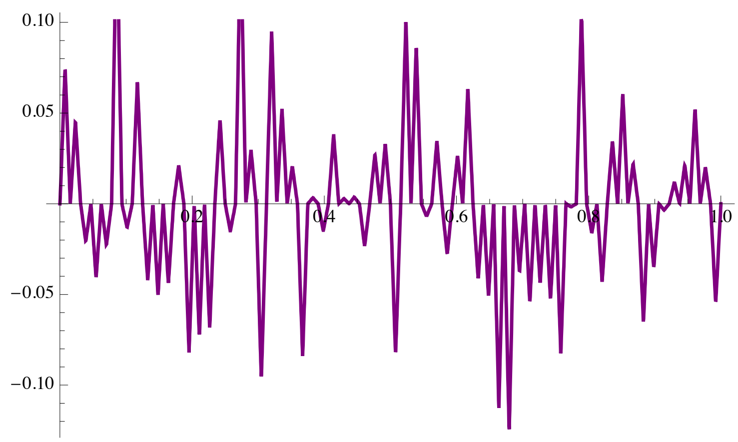

Figure 10: A sample path of the random function $V_6(t)$.

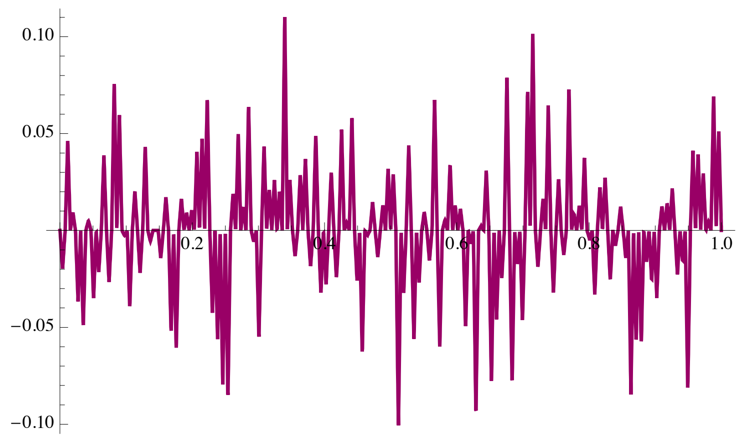

Figure 11: A sample path of the random function $V_7(t)$.

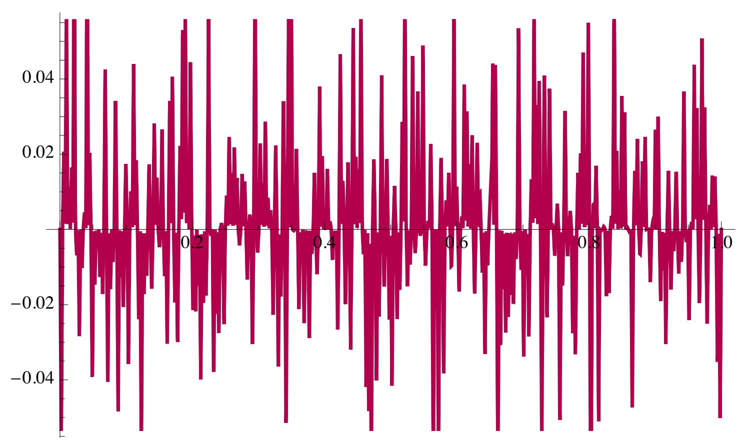

Figure 12: A sample path of the random function $V_8(t)$.

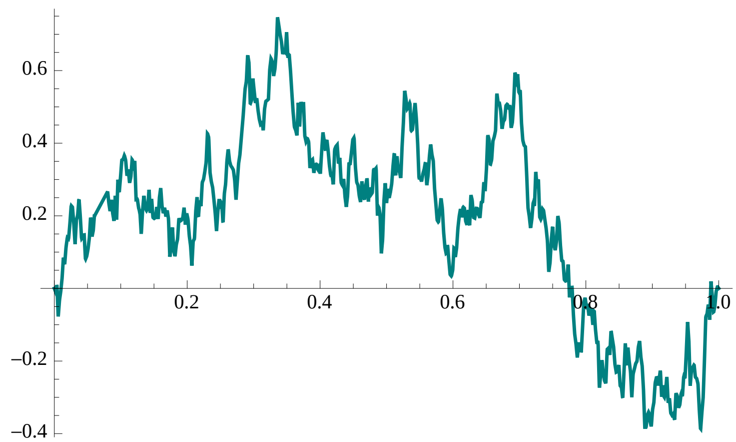

Figure 13: A sample path of the random function $U_8(t)$.

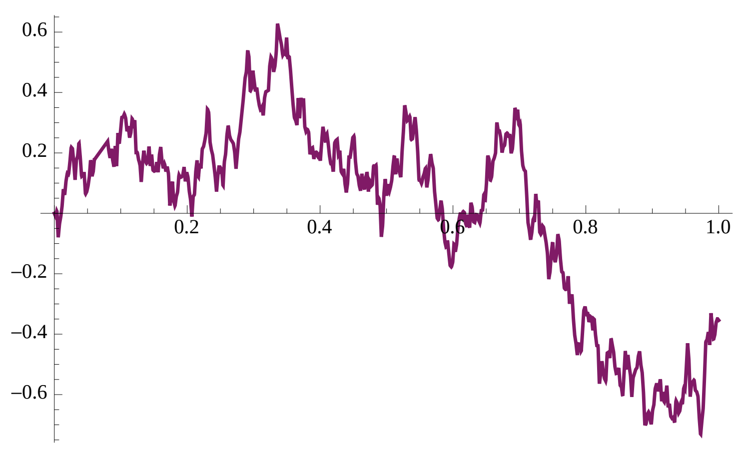

Figure 14: A sample path of the random function $S_8(t)$.