Review Handout 3: Poisson Processes

Math/Stat 491: Introduction to Stochastic Processes

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Part 1: Exponential distribution

1. T has an exponential distribution with parameter λ, and we write $T \sim$ exponential(λ), if T has distribution function F_T given by $1 - F_T(t) = \exp(-\lambda t)$ for $t \geq 0$. Thus T has density function $f_T(t) = \lambda \exp(-\lambda t) 1_{[0,\infty)}(t)$ and hazard function

$$\lambda(t) \equiv \frac{f_T(t)}{1 - F_T(t)} = \lambda, \qquad t \geq 0.$$

2. If $T \sim \exp(\lambda)$, then $E(T) = 1/\lambda$, $Var(T) = 1/\lambda^2$, and T has moment generating function $\phi_X(s) = Ee^{sT} = \lambda/(\lambda - s)$ for $s < \lambda$.

3. The exponential distribution has the "lack of memory" property:

$$P(T > s + t | T > s) = P(T > t) \quad \text{for all} \ \ s, \ t \geq 0.$$

4. If $S \sim \exp(\lambda)$ and $T \sim \exp(\mu)$ are independent, then:
   - $\min\{S, T\} \sim \exp(\lambda + \mu)$.
   - $P(S \leq T) = \lambda/(\lambda + \mu)$. (Proof:

$$\begin{aligned} P(S \leq T) &= E\{P(S \leq T|S)\} = E\{\exp(-\mu S)\} \\ &= \int_0^\infty e^{-\mu s} \lambda e^{-\lambda s} ds = \frac{\lambda}{\lambda + \mu} \int_0^\infty (\lambda + \mu) e^{-(\lambda+\mu)s} ds \\ &= \frac{\lambda}{\lambda + \mu}. \end{aligned}$$

5. If $T_j \sim \exp(\lambda_j)$, $j = 1, \ldots, m$ are independent, then:
   - $\min_{1 \leq j \leq m} T_j \sim \exp(\lambda_1 + \cdots + \lambda_m)$.
   - $P(T_j = \min_{1 \leq k \leq m} T_k) = \lambda_j/(\lambda_1 + \cdots + \lambda_m)$.

- If I satisfies $T_I = \min_{1 \le k \le m} T_k$ and $V \equiv \min_{1 \le k \le m} T_k$, then I and V are independent.

6. If $\tau_1, \ldots, \tau_m$ are independent exponential(λ) random variables, then $T_m \equiv \tau_1 + \cdots + \tau_m \sim$ Gamma(m, λ): that is, T_m has density function f_{T_m} given by

$$f_{T_m}(t) = \frac{(\lambda t)^{m-1}}{(m-1)!} \lambda e^{-\lambda t} 1_{[0,\infty)}(t).$$

If $T \sim$ Gamma(r, λ) with $r > 0$ and $\lambda > 0$, then T has density

$$f_T(t) = \frac{(\lambda t)^{r-1}}{\Gamma(r)} \lambda e^{-\lambda t} 1_{[0,\infty)}(t)$$

where $\Gamma(r) \equiv \int_0^\infty v^{r-1} e^{-v} dv$ is the *Gamma function*. Note that $\Gamma(r) = (r-1)\Gamma(r-1)$ and for $r > 1$, and hence $\Gamma(m) = (m-1)!$ for $m \in \{1, 2, \ldots\}$.

Part 2: Poisson processes

1. **Definition 1**. Let $\tau_1, \tau_2, \ldots$ be independent and identically distributed exponential(λ) random variables, and let $T_n \equiv \tau_1 + \cdots + \tau_n$ for $n \ge 1$, $T_0 \equiv 0$. Then the process

$$N(t) = \max\{n \ge 1 : \ T_n \le t\} = \sum_{n=0}^{\infty} 1_{[T_n \le t]}$$

is a *Poisson process with rate* λ.

2. If $X \sim$ Poisson(λ), then $P(X = k) = e^{-\lambda} \lambda^k / k!$ for $k \in \{0, 1, 2, \ldots\}$, $E(X) = \lambda$, $Var(X) = \lambda$, $E\{X(X-1)\cdots(X-k+1)\} = \lambda^k$, and X has moment generating function $\phi_X(s) = E(e^{sX}) = \exp(\lambda(e^s - 1))$.

3. Properties of $N(t)$:

   - $P(N(t) = 0) = P(T_1 > t) = P(\tau_1 > t) = \exp(-\lambda t)$ for $t > 0$.
   - $N(t) \sim$ Poiss(λt); that is,

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t).$$

- $E\{N(t)\} = \lambda t$; $Var(N(t)) = \lambda t$.
- The process $\{N(s+t) - N(s) : \ t \geq 0\}$ is independent of $\{N(r) : 0 \leq r \leq s\}$.
- $N(t)$ has independent increments: for any points $0 = t_0 < t_1 < \cdots < t_k$, the random variables

  $$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_k) - N(t_{k-1}) \quad \text{are independent.}$$

4. **Theorem.** $\{N(t) : \ t \geq 0\}$ is a Poisson process with rate λ if and only if the following three conditions hold:
(i) $N(0) = 0$.
(ii) $N(t+s) - N(s) \sim \text{Poiss}(\lambda t)$.
(iii) $N(t)$ has independent increments.

5. $N(t)$ has moment generating function

$$\phi_{N(t)}(s) = Ee^{sN(t)} = \exp(\lambda t(e^s - 1)).$$

6. $\{M(t) : t \geq 0\}$ defined by $M(t) \equiv N(t) - \lambda t$ is a martingale: $E\{M(t)|\mathcal{F}_s\} = M(s)$ where $\mathcal{F}_s \equiv \sigma\{N(r) : \ 0 \leq r \leq s\}$.

7. If $S_n \sim \text{Binomial}(n, p_n)$ and $np_n \to \lambda > 0$, then

$$P(S_n = k) = \binom{n}{k} p_n^k (1 - p_n)^{n-k} \to \frac{\lambda^k}{k!} \exp(-\lambda).$$

8. If $X_1, \ldots, X_n$ are independent with $X_i \sim \text{Bernoulli}(p_i)$, and $S_n = \sum_{i=1}^n X_i$, and $T_n \sim \text{Poiss}(\sum_{i=1}^n p_i)$, then for any subset A of $\{0, 1, 2, \ldots\}$,

$$|P(S_n \in A) - P(T_n \in A)| \leq \sum_{i=1}^n p_i^2 \leq \max_{1 \leq i \leq n} p_i \cdot \sum_{i=1}^n p_i.$$

Note that when $p_1 = \cdots = p_n = p_{0,n}$, the bound becomes $np_{0,n}^2 = (np_{0,n})^2/n \approx \lambda^2/n$ if $np_{0,n} \to \lambda$.

9. **Definition**. $\{N(t) : \ t \geq 0\}$ is a *non-homogeneous Poisson process* with rate function $\lambda(s) \geq 0$ if:
(i) $N(0) = 0$;
(ii) $N(t+s) - N(s) \sim \text{Poiss}\left(\int_s^{s+t} \lambda(v) dv\right)$;
(iii) N has independent increments.

10. Properties of a non-homogeneous Poisson process N:

   - $E\left(N(t)\right) = \int_0^t \lambda(v)dv \equiv \mu(t)$ where $\mu(s) \le \mu(t)$ for $s \le t$.

   - If N is a standard Poisson process with rate 1 and μ is a non-decreasing differentiable function, then $\widetilde{N}(t) \equiv N(\mu(t))$ is a non-homogeneous Poisson process with rate function $\mu'(t) \equiv \lambda(t)$.
   - If $\tau_1 = \inf\{t \ge 0 : N(t) = 1\}$, then

$$\begin{aligned} 1 - F_{\tau_1}(t) &= P(\tau_1 > t) = P(N(t) = 0) = \exp\left(-\int_0^t \lambda(v)dv\right) = \exp(-\mu(t)), \\ f_{\tau_1}(t) &= \mu'(t)\exp(-\mu(t)) = \lambda(t)\exp(-\mu(t)), \\ \lambda_{\tau_1}(t) &= \frac{f_{\tau_1}(t)}{1 - F_{\tau_1}(t)} = \lambda(t). \end{aligned}$$

11. **Definition**. Let $Y_1, Y_2, Y_3, \ldots$ be independent and identically distributed random variables with $E(Y_i) = \mu$ and $Var(Y_i) = \sigma^2$. Let $\{N(t) : t \ge 0\}$ be a Poisson process with rate λ. Then the process $\{S_t : \ t \ge 0\}$ defined by

$$S_t \equiv Y_1 + \cdots + Y_{N(t)} \quad \text{for} \quad t \ge 0,$$

   is a *compound Poisson process*.

12. **Theorem.** Suppose that N is an integer-valued random variable independent of $Y_1, Y_2, \ldots$, and let $S \equiv \sum_{i=1}^N Y_i$. Then:
   (i) If $E|Y_i| < \infty$ and $E(N) < \infty$, then $E(S) = E(N) \cdot E(Y) = E(N) \cdot \mu$ where $\mu = E(Y_1)$.
   (ii) If $E(Y_1^2) < \infty$ and $Var(N) < \infty$, then
   $Var(S) = E(N)\sigma^2 + Var(N)\mu^2$ where $\sigma^2 = Var(Y_1)$.
   (iii) If $N \sim$ Poiss(λ) and $E(Y_1^2) < \infty$, then $E(S) = \lambda\mu$ and $Var(S) = \lambda E(Y_1^2) = \lambda(\sigma^2 + \mu^2)$.
   (iv) If N is replaced by $N(t)$ for a Poisson process with rate λ, then $E(S_t) = \lambda t\mu$ and $Var(S_t) = \lambda t E(Y_1^2) = \lambda t(\sigma^2 + \mu^2)$.

13. **Thinning of a Poisson process.** Suppose that $Y_1, Y_2, \ldots$ are independent and identically distributed integer-valued random variables, and suppose that $\{N(t) : t \ge 0\}$ is a Poisson process with rate λ independent of the Y_i's. Define new processes $N_j(t)$ by

$$N_j(t) = \#\{i \le N(t) : \ Y_i = j\}, \quad t \ge 0, \quad j = 1, 2, 3, \ldots, k.$$

Theorem. $N_1, N_2, \ldots, N_k$ are independent Poisson processes with rates $\lambda P(Y_1 = j)$, $j = 1, \ldots, k$.

14. **Superposition of independent Poisson processes.** Suppose that $N_1, N_2, \ldots$ are independent Poisson processes with rates $\lambda_1, \lambda_2, \ldots$.
Theorem. If $N_1, N_2, \ldots, N_k$ are independent Poisson processes with rates $\lambda_1, \ldots, \lambda_k$, then $N(t) \equiv N_1(t) + \cdots + N_k(t)$ is a Poisson process with rate $\lambda_1 + \cdots + \lambda_k$.

15. **Conditioning Poisson processes.** Let $T_1, T_2, \ldots, T_n$ be the arrival times of a Poisson process with rate λ. Let $U_1, \ldots, U_n$ be i.i.d. Uniform$[0,t]$ random variables, and let $0 \le U_{(1)} \le \cdots \le U_{(n)} \le t$ be the *order statistics* of the U_i's.
Theorem.

$$((T_1, \ldots T_n)|N(t) = n) \stackrel{d}{=} (U_{(1)}, \ldots, U_{(n)})$$

and

$$(N(s),\ 0 \le s \le t|N(t) = n) \stackrel{d}{=} \left(\sum_{i=1}^{n} 1_{[U_{(i)} \le s]},\ 0 \le s \le t \right),$$

where

$$N_n(s) = \sum_{i=1}^{n} 1_{[U_{(i)} \le s]} = \sum_{i=1}^{n} 1_{[U_i \le s]}.$$

Thus if $0 = t_0 < t_1 < \cdots < t_k = t$, then

$$\begin{aligned}
&(N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_k) - N(t_{k-1})|N(t) = n) \\
&\stackrel{d}{=} \ (N_n(t_1) - N_n(t_0), N_n(t_2) - N_n(t_1), \ldots, N_n(t_k) - N_n(t_{k-1})) \\
&\sim \ \text{Multinomial}_k(n, (t_1 - t_0, t_2 - t_1, \ldots, (t_k - t_{k-1}))/t).
\end{aligned}$$

16. **Poissonization:**

- (HW #2, problem 6). If $X_1, X_2, \ldots$ are i.i.d. Bernoulli(p) and $N \sim$ Poiss(λ), then

$$S_N = \sum_{i=1}^{N} X_i \sim \text{Poiss}(\lambda p).$$

- (HW #3, problem 7). If $\underline{S}_n = (S_{n,1}, \ldots, S_{n,k}) = \sum_{i=1}^n \underline{X}_i \sim \text{Mult}_k(n, (p_1, \ldots, p_k))$ where $\underline{X}_i \sim \text{Mult}_k(1, (p_1, \ldots, p_k))$, and $N \sim \text{Poiss}(\lambda)$ is independent of $\underline{S}_n$, then

$$\underline{S}_N \stackrel{d}{=} (Y_1, \ldots, Y_k)$$

where $Y_1, \ldots, Y_k$ are independent and $Y_j \sim \text{Poiss}(\lambda p_j)$.

- If $N_n(t) = \sum_{i=1}^n 1_{[U_i \le t]}$ where $U_1, U_2, \ldots$ are i.i.d. Uniform$(0, 1)$, and $N_\lambda \sim \text{Poiss}(\lambda)$ is independent of the U_i's, then $\mathbb{N}(t) \equiv N_{N_\lambda} = \sum_{i=1}^{N_\lambda} 1_{[U_i \le t]}$ is a Poisson process on $[0, 1]$ with rate λ.