Probability Basics: Handout 2

Math/Stat 394: Probability I

Wellner; 1/14/2000

Normal Random Variables and Distributions:
We say that $Z \sim N(0,1)$ if

$$f_Z(z) = (2\pi)^{-1/2} \exp(-z^2/2) \equiv \phi(z)$$

for $-\infty < z < \infty$.
We will soon show that $\mu_Z = 0$ and $\sigma_Z = 1$.
If $Z \sim N(0,1)$, then it has distribution function (c.d.f.)

$$F_Z(z) = P(Z \le z) = \int_{-\infty}^{z} \phi(t)dt \equiv \Phi(z)\,.$$

Let $X \equiv \mu + \sigma Z$, with $\sigma > 0$; then $\mu_X = \mu$ and $\sigma_X = \sigma$. Furthermore

$$F_X(x) = P(X \le x) = P(\mu + \sigma Z \le x) = P(Z \le \frac{x-\mu}{\sigma}) = \Phi(\frac{x-\mu}{\sigma})\,,$$

and

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) = \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)\,. \end{aligned}$$

This is the $N(\mu, \sigma^2)$ density, and we will say that $X \sim N(\mu, \sigma^2)$.

Let $X \sim N(\mu, \sigma^2)$. Then

$$P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P\left(Z \le \frac{x-\mu}{\sigma}\right)\,.$$

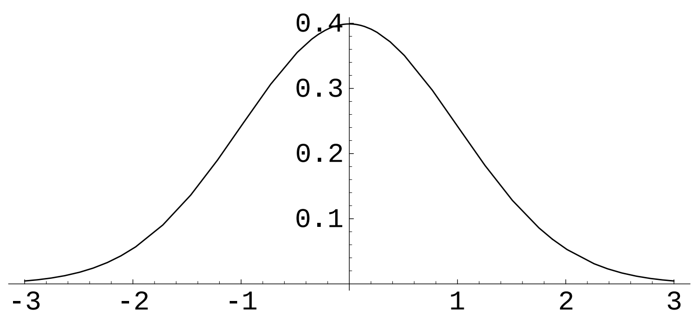

Figure 1:Plot of standard normal density $\phi(z)$.

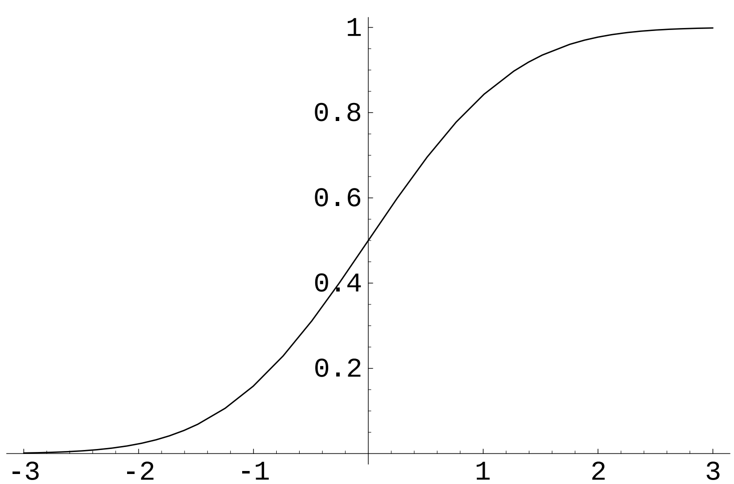

Figure 2: Plot of standard normal distribution function $\Phi(z)$.

Let $X \sim N(100, (16)^2)$ denote IQ. Then

$$P(68 \le X \le 116) = P\left(\frac{68-100}{16} \le \frac{X-100}{16} \le \frac{116-100}{16}\right) = P(-2 \le Z \le 1) = .818\,.$$

Let $X \sim N(68.5, 2.25^2)$ represent the height of adult males. Then

$$P(X \le 73.0) = P\left(\frac{X-68.5}{2.25} \le \frac{73-68.5}{2.25}\right) = P(Z \le 2) = .978\,.$$

Main use: CLT. Let X denote the numerical value of a random variable in any basic experiment; call its mean μ and standard deviation σ. Let $X_1, \ldots, X_n$ denote independent replications of the X experiment. Then

$$T_n \equiv (\text{the total sum}) \equiv X_1 + \cdots + X_n$$

has mean $\mu_n = n\mu$ and standard deviation $\sigma_n = \sqrt{n}\sigma$. Also

$$\overline{X}_n \equiv (\text{the average}) \equiv \frac{1}{n}(X_1 + \cdots + X_n) = T_n/n$$

has mean μ and standard deviation $\sigma/\sqrt{n}$, and

$$Z_n \equiv \frac{T_n - n\mu}{\sqrt{n}\sigma} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$$

has mean 0 and standard deviation 1.
CLT: Z_n is approximately $N(0,1)$. That is,

$$P(a \le Z_n \le b) \to P(a \le Z \le b)$$

for all a, b where $Z \sim N(0,1)$.