Handout 5: THE POISSON PROCESS, Continued
Math/Stat 394: Probability I
Wellner; 1/28/2000

Example 1. Each August shooting stars (the Perseids) appear at the rate $\nu = .5$ per minute. Suppose we start watching the meteor shower at midnight on a particular night. Then let

$$\mathbb{N}(t) = (\text{ \# shooting stars in t minutes }) \sim Poisson(\lambda = \nu t)\,.$$
$$Y \equiv (\text{time until next sighting}) \sim Exponential(\nu = .5)\,.$$

(a) We start watching at midnight; $\mathbb{N}(2)$ is the number we see by 12:02. Now $\mathbb{N}(2) \sim Poisson(\lambda = \nu t = (.5)(2) = 1)$. Thus the chance that we see at least one shooting star in this period is

$$P(\mathbb{N}(2) \geq 1) = 1 - P(\mathbb{N}(2) = 0) = 1 - e^{-1}\frac{1^0}{0!} = 1 - e^{-1} = .632\,.$$

(b) The probability we will see exactly 3 shooting stars by 12:04 is, since $\mathbb{N}(4) \sim Poisson(\lambda = \nu t = (.5)(4) = 2)$,

$$P(\mathbb{N}(4) = 3) = e^{-2}\frac{2^3}{3!} = .180\,.$$

(c) The number $\mathbb{N}(60)$ we observe in an hour (= 60 minutes) is $\mathbb{N}(60) \sim Poisson(\lambda = \nu t = (.5)(60) = 30)$, and this is approximated by a Normal distribution with the same mean and variance, namely $\mathbb{N}(30, 30)$. (Recall that if $X \sim Poisson(\lambda)$, we showed that $E(X) = \lambda$ and $Var(X) = \lambda$.) Thus the probability that we will see at least 25 shooting stars by 1:00 AM is

$$\begin{aligned} P(\mathbb{N}(60) \geq 25) &= P(\mathbb{N}(60) \geq 24.5) \\ &= P\left(\frac{\mathbb{N}(60) - 30}{\sqrt{30}} \geq \frac{24.5 - 30}{\sqrt{30}}\right) \\ &\doteq P(Z \leq -1.004) = .842\,. \end{aligned}$$

The exact probability is

$$\begin{aligned} P(\mathbb{N}(60) \geq 25) &= 1 - P(\mathbb{N}(60) \leq 24) \\ &= 1 - \sum_{k=0}^{24} e^{-30}\frac{(30)^k}{k!} \\ &= 1 - .15724 = 0.84276\,. \end{aligned}$$

(d) The time W_{30} it takes us to see 30 shooting stars is

$$W_{30} = Y_1 + \cdots + Y_{30} \sim Gamma(r = 30, \nu = .5)\,,$$

and this can be approximated by a normal distribution with the same mean and variance, namely

$$N(\mu, \sigma^2) = N(r/\nu, r/\nu^2) = N(60, 120)\,,$$

since $r = 30$ and $\nu = .5$. (Looking at our graph of W_r for $r = 32$ and $\nu = 1$, we might guess that this approximation is ok, but not great.) To compute the probability that the time we take to observe 30 shooting stars is more than 54 minutes, one approach is to use this normal approximation: thus

$$P(W_{30} > 54) = P\left(\frac{W_{30} - 60}{\sqrt{120}} > \frac{54 - 60}{\sqrt{120}}\right) \doteq P(Z > -.5477) = .708\,.$$

Another approach is to use the fundamental identity:

$$P(W_{30} > 54) = P(\mathbb{N}(54) < 30) = P\left(\frac{\mathbb{N}(54) - 27}{\sqrt{27}} < \frac{29.5 - 27}{\sqrt{27}}\right) \doteq P(Z < .481) = .689\,.$$

A look at our pictures of Poisson distributions makes us believe this approximation is closer. In fact, the exact true value is

$$P(\mathbb{N}(54) \leq 29) = \sum_{k=0}^{29} \exp(-27)\frac{27^k}{k!} = .6935\,.$$

I obtained this value using Mathematica.

(e) Let Z_2 denote the time from our 1st sighting until our third sighting. Thus $Z_2 = Y_2 + Y_3$ has the same distribution as W_2, namely Gamma$(2, \nu = .5)$ with density

$$f_{Z_2}(t) = f_{W_2}(t) = \frac{t}{4}e^{-t/2} \qquad \text{for} \qquad t > 0\,.$$

Thus

$$P(Z_2 > 3) = \int_3^\infty (t/4)e^{-t/2} = -\{(t/2)e^{-t/2} + e^{-t/2}\}|_3^\infty = e^{-3/2}\{\frac{3}{2} + 1\} = .558\,.$$

(f) Suppose that we see 8 shooting stars in the first 20 minutes. What is the probability that exactly 3 are seen in the first 5 minutes? According to Poisson fact 1, this number $N(5)$ conditional on $N(20) = 8$, has a Binomial$(n = 8, p = 5/20 = .25)$ distribution:

$$(N(5)|N(20) = 8) \sim \text{Binomial}(8, 25)\,.$$

Thus

$$P(N(5) = 3|N(20) = 8) = \binom{8}{3}(.25)^3(.75)^{8-3} = .208\,.$$

(g) Suppose we start watching at midnight. Given that we wated more than 5 minutes to see the first shooting star, what is the probability that we waited more than 8 minutes?

$$P(Y_1 > 8|Y_1 > 5) = P(Y_1 > 3) = e^{-\nu t}|_{t=3} = e^{-3/2} = .223$$

by lack of memory.

(h) Suppose that we will get bored and quit watching if we have to wait more than 3 minutes between sightings. Well,

$$\begin{aligned} p &= P(Y_1 < 3)P(Y_2 < 3)P(Y_3 < 3)P(Y_4 < 3)P(Y_5 < 3)P(Y_6 > 3) \\ &= P(Y_1 < 3)^5 P(Y_1 > 3) = .777^5 \times .223 = .0632 \end{aligned}$$

by using the probability we computed in (g).