Handout 7: Math/Stat 394: Probability I
Joint Probability Distributions, Continuous Case
Wellner; 2/16/2000

Joint Probability Distributions, Continuous Case: Suppose that X, Y are two random variables defined on the same sample space. Then the *joint density function* $f_{X,Y}$ of (X, Y) is a function satisfying the following properties:

(i) $f_{X,Y}(x, y) \geq 0$ for all x, y.

(ii) $\int \int f_{X,Y}(x, y) dx dy = 1$.

(iii) $P((X, Y) \in A) = \int \int_A f_{X,Y}(x, y) dx dy$.

The *marginal probability density functions* f_X and f_Y of X and Y respectively, are obtained from the joint mass function by integratinging out:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy \qquad \text{and} \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx.$$

The *joint distribution function* (or *bivariate CDF*) $F_{X,Y}$ of (X, Y) is the function

$$F_{X,Y}(x, y) \equiv P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x', y') dx' dy' .$$

Note that from the definition of $F_{X,Y}$ it follows that we can recover the joint density function $f_{X,Y}$ in the continuous case by differentiating the joint distribution function $F_{X,Y}$:

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y) .$$

Furthermore, we can recover the marginal distribution functions F_X and F_Y of X and Y respectively from the joint distribution function $F_{X,Y}$:

$$F_X(x) = P(X \leq x) = P(X \leq x, Y \leq \infty) = F_{X,Y}(x, \infty) ,$$

and

$$F_Y(y) = P(Y \leq y) = P(X \leq \infty, Y \leq y) = F_{X,Y}(\infty, y) .$$

Analogously to the definition of conditional mass functions in the discrete case, we define $f_{Y|X}$, the *conditional density of* Y *given* X, and $f_{X|Y}$, the *conditional density of* X *given* Y, by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \qquad \text{if } \; f_X(x) > 0$$

and

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{if } \; f_Y(y) > 0\,.$$

The *conditional mean functions* (or *regression functions*) $E(Y|X=x)$ and $E(X|Y=y)$ are defined simply as the (conditional) expectations with respect to the conditional densities $f_{Y|X}$ and $f_{X|Y}$ respectively:

$$m(x) \equiv E(Y|X=x) = \int y f_{Y|X}(y|x)dy$$

and

$$E(X|Y=y) = \int x f_{X|Y}(x|y)dx\,.$$

Example 1. Suppose that X and Y have a uniform density on the triangle $T \equiv \{(x,y): \;\; 0 \le y \le x \le 1\}$. Since the area of T is $1/2$, we find that

$$f_{X,Y}(x,y) = 2 \cdot 1_T(x,y)$$

See Figures 1 and 2. Recall that

$$1_T(x,y) = \begin{cases} 1, & (x,y) \in T, \\ 0, & (x,y) \notin T\,. \end{cases}$$

We proceed to find the corresponding marginal densities f_X, f_Y, the conditional densities $f_{Y|X}$, $f_{X|Y}$, and the joint distribution function $F_{X,Y}$. First the marginal densities:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^x 2dy = 2x$$

if $0 \le x \le 1$; otherwise the marginal density is 0. Thus $f_X(x) = 2x1_{[0,1]}(x)$. Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_y^1 2dx = 2(1-y)$$

if $0 \le y \le 1$; otherise the marginal density of Y is 0. Hence $f_Y(y) = 2(1-y)1_{(0,1)}(y)$. Now we can compute the conditional densities:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2}{2x} \qquad \text{if } \ 0 \le y \le x \le 1\,.$$

Thus we have

$$f_{Y|X}(y|x) = \frac{1}{x} 1_{(0,x)}(y)\,;$$

i.e. $(Y|X=x) \sim \text{Uniform}(0,x)$. Similarly,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} \qquad \text{if } \ 0 \le y \le x \le 1\,.$$

Thus we have

$$f_{X|Y}(x|y) = \frac{1}{1-y} 1_{(y,1)}(x)\,;$$

i.e. $(X|Y=y) \sim \text{Uniform}(y,1)$. As a consequence, it is easy to compute the conditional mean functions:

$$E(Y|X=x) = \int y f_{Y|X}(y|x)dy = \int_0^x y\frac{1}{x}dy = \frac{x}{2}\,,$$

and

$$E(X|Y=y) = \int x f_{X|Y}(x|y)dx = \int_y^1 x\frac{1}{1-y}dx = \frac{1+y}{2}\,.$$

See figure 1 for a plot of $E(Y|X=x)$. Finally the joint distribution function $F_{X,Y}$ is computed as follows: for $(x,y) \in T$ where T is the triangle where the joint density is positive,

$$\begin{aligned} F_{X,Y}(x,y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x',y')dx'dy' \\ &= \int_0^x \int_0^y 2 \cdot 1_T(x',y')dx'dy' \\ &= 2((y^2/2 + y(x-y)) = 2xy - y^2 \end{aligned}$$

by decomposing the region into the triangle with base of length y and height y, and the remaining square with area $y(x-y)$. Note that for other values of (x, y) we can easily compute the joint distribution function $F_{X,Y}$ from its values on the boundary of the triangle T: for $0 < x < y < \infty$ we have

$$F_{X,Y}(x, y) = P(X \le x, Y \le y) = P(X \le x, Y \le x) = F_{X,Y}(x, x) = x^2 \,,$$

and for $1 < x < \infty$, $0 < y \le 1$,

$$F_{X,Y}(x, y) = P(X \le x, Y \le y) = P(X \le 1, Y \le y) = F_{X,Y}(1, y) = 2y - y^2 \,.$$

On the other hand, for $1 < x < \infty$ and $1 < y < \infty$,

$$F_{X,Y}(x, y) = P(X \le x, Y \le y) = P(X \le 1, Y \le 1) = F_{X,Y}(1, 1) = 2-1^2 = 1 \,.$$

See Figure 3 for a plot of $F_{X,Y}$ (and note that the marginal distribution functions F_X and F_Y do appear in the plot). Note that we do recover the marginal distribution functions:

$$F_X(x) = x^2 = F_{X,Y}(x, \infty) = F_{X,Y}(x, x)$$

for $0 \le x \le 1$ and

$$F_Y(y) = 1 - (1 - y)^2 = 2y - y^2 = F_{X,Y}(\infty, y)$$

for $0 \le y \le 1$.

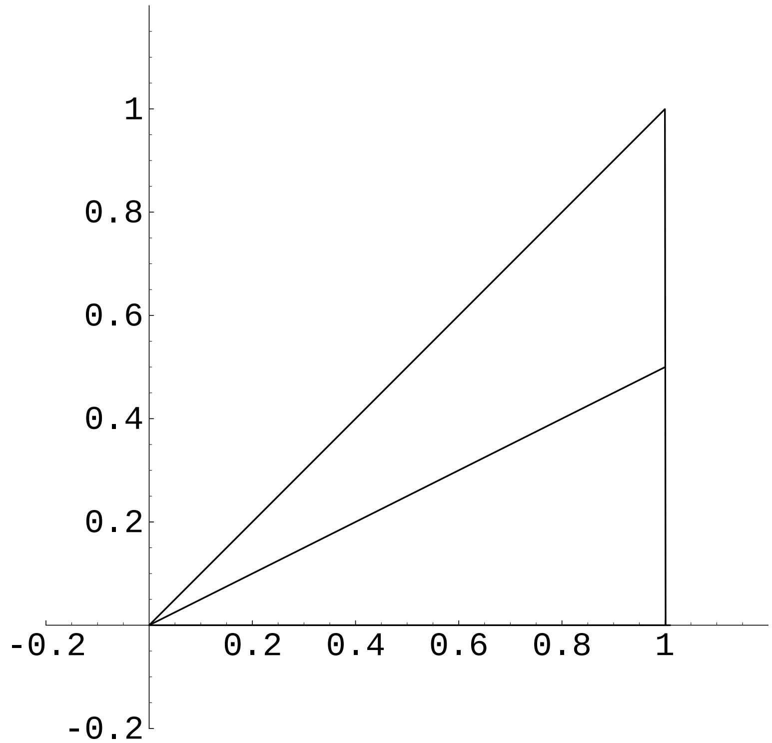

Figure 1: Plot of Triangle T and $E(Y|X = x)$.

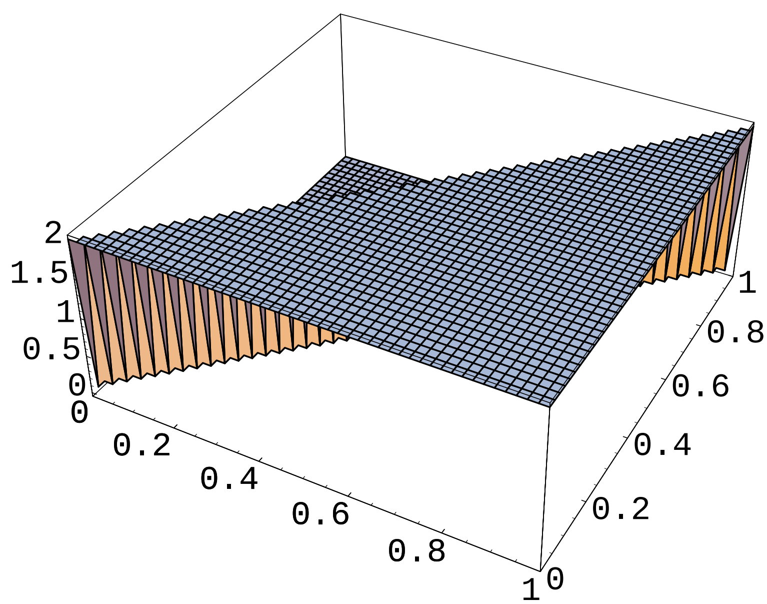

Figure 2: Plot of joint density $f_{X,Y}$.

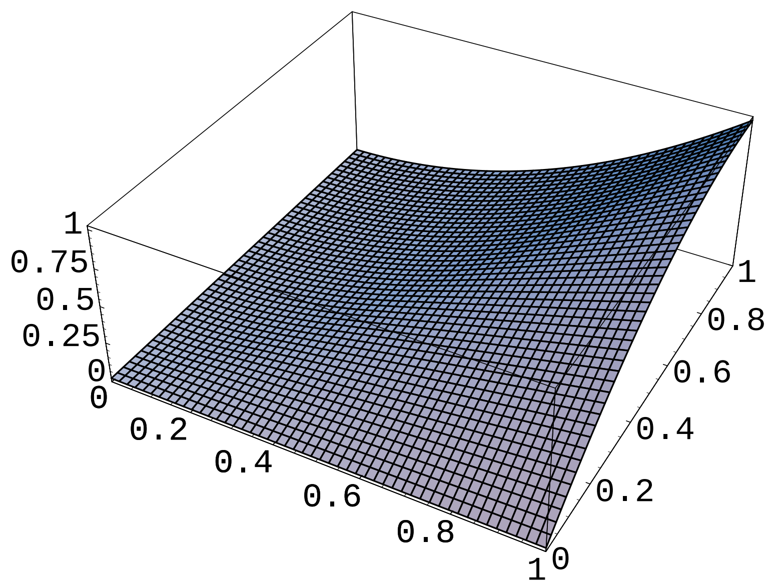

Figure 3: Plot of joint distribution function $F_{X,Y}$.