Statistics 492, Problem Set 3

Wellner; 1/21/14

Reading: Karlin and Taylor, chapter 7, pages 340-365
Karlin and Taylor, chapter 6, section 8, pages 313-325.
Klebaner, chapter 4, pages 87-109,
Klebaner, chapter 8, pages 169-177.

Reminder: Preliminary project titles and descriptions due Thursday, 23 January

Due: Tuesday, 28 January 2014.

1. Klebaner, Exercise 3.11, page 85: Let B be a standard Brownian motion. Show that the following processes are Brownian motions on $[0, T]$:
(i) $T < \infty$: $X(t) = B(T - t) - B(T)$.
(ii) $T = \infty$: $Y(t) = cB(t/c^2)$ where $c \in \mathbb{R}$.

2. Let $B_\mu(t) \equiv B(t) + \mu t$ where $\mu \in \mathbb{R}$ and B is standard Brownian motion. The process B_μ is Brownian motion with drift μ. For $a > 0$, $b > 0$, let $\tau_{a,b} \equiv \min\{T_{-a}, T_b\}$ where $T_{-a} \equiv \inf\{t > 0 : B_\mu(t) = -a\}$, $T_b \equiv \inf\{t > 0 : B_\mu(t) = b\}$.
(i) Find $P_0(\tau_{a,b} = T_{-a}) = P_0(T_{-a} < T_b)$ and $P_0(\tau_{a,b} = T_b) = P_0(T_b < T_{-a})$.
(ii) Find $E_0\tau_{a,b}$.
(iii) When $\mu < 0$ and $a = \infty$ (so $-a = -\infty$), find $P_0(T_b < \infty)$.

3. Let $B_\mu(t) \equiv B(t) - \mu t$ where $\mu > 0$ and B is standard Brownian motion. The process B_μ is Brownian motion with (negative) drift. Let $\tau = \inf\{t > 0 : B_\mu(t) = b > 0\}$.
(i) Use the martingale $Y_c(t) = \exp(cB(t) - c^2t/2)$ with $c = \mu + (\mu^2 + 2\lambda)^{1/2}$ to show that
$$E_0 \exp(-\lambda\tau) = \exp(-b(\mu + (\mu^2 + 2\lambda)^{1/2})).$$
(ii) Let $\lambda \to 0$ to show that $P_0(\tau < \infty) = \exp(-2b\mu)$.
(iii) Interpret your results in terms of pictures for both B and B_μ.

4. Prove Kolmogorov's inequality for Brownian motion:
$$P(\sup_{0 \le s \le t} |B(s)| > \epsilon) \le t/\epsilon^2.$$

5. Use the result of the previous exercise to prove that $t^{-1}B(t) \to_{a.s.} 0$. Hint: Take $\epsilon = 2^{2n/3}$ and $t = 2^n$; then apply the Borel-Cantelli lemma: if $\sum_{n=1}^\infty P(A_n) < \infty$, then $P(A_n \text{ infinitely often}) = 0$.

6. **Optional bonus problem:** Show that $Y_t = (1+t)^{-1/2} \exp(B_t^2/2(1+t))$ is a martingale and use this to show that $\limsup_{t\to\infty} B_t/((1+t)\log(1+t))^{1/2} \leq 1/\sqrt{2}$ almost surely.