Statistics 492, Problem Set 4

Wellner; 1/28/14

Reading:	Klebaner, chapter 4, pages 102-116, Klebaner, chapter 7, pages 169-177.
Reminder:	No Problem Set 5 next week; Problems Set 5 will be handed out on Tuesday 18 February. Lectures by Shuliu Yuan on February 4, 6, 11 will cover renewal theory $\cdots$ based on Durrett, chapter 3; Karlin and Taylor, chapter 5 Please work on your projects during the next two weeks.
Due:	Tuesday, 4 February 2014.

1. Klebaner, Exercise 4.1, page 114: Let $\tau_1 < \tau_2$ be stopping times with respect to the natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$ of Brownian motion B on $[0,T]$. Show that $X(t) = 1_{(\tau_1,\tau_2]}(t)$ is a simple predictable process.

2. Klebaner, Exercise 4.5, page 115: Show that if $X(t,s)$ is a non-random function of both s and t with $\int_0^t X^2(t,s)ds < \infty$, then $Y(t) = \int_0^t X(t,s)dB(s)$ is a Gaussian random variable $Y(t)$, and the process $\{Y(t) : 0 \le t \le T\}$ is a Gaussian process with zero mean and covariance function given by $Cov(Y(t), Y(t+v)) = \int_0^t X(t,s)X(t+v,s)ds$ for $v \ge 0$.

3. Let B be standard Brownian motion and let $x \in \mathbb{R}$. Define a new process Y_t by

$$Y_t = e^{-t/2}x + e^{-t/2}\int_0^t e^{s/2}dB_s.$$

(i) Show that $\{Y_t : \ t \ge 0\}$ is a Gaussian process with mean $e^{-t/2}x$ and variance $1 - e^{-t}$.
(ii) Now let $Z \sim N(0,1)$ be independent of B and define $\tilde{Y}(t)$ by

$$\tilde{Y}_t = e^{-t/2}Z + e^{-t/2}\int_0^t e^{s/2}dB_s.$$

Show that $\tilde{Y}$ is a mean 0 Gaussian process with variance 1, and show that $Y(t) \stackrel{d}{=} e^{-t/2}x + e^{-t/2}B(e^t - 1)$. Thus with $Z(t) \equiv \int_0^t e^{s/2}dB(s)$, $Z(t) \stackrel{d}{=} B(e^t - 1)$ as processes, or, equivalently, $Z(\log(t+1)) \stackrel{d}{=} B(t)$.
(iii) Compute the covariance of the process $\tilde{Y}$. The processes Y and $\tilde{Y}$ are known as *Ornstein-Uhlenbeck processes.*

4. Let X_n be a sequence of random variables with normal distributions $N(\mu_n, \sigma_n^2)$ and suppose that $X_n \to_d X$. (i) Show that the distribution of X is either normal or degenerate (i.e. $P(X = x_0) = 1$ for some $x_0 \in \mathbb{R}$).
(ii) Show that if $E(X_n) \to \mu$ and $Var(X_n) \to \sigma^2 > 0$, then the limiting distribution (of X) is $N(\mu, \sigma^2)$.
(iii) Since convergence in probability implies convergence in distribution, deduce convergence of Itô integrals of simple deterministic processes to a Gaussian limit.

5. Use the Itô isometry to calculate the variances of

$$\int_0^t |B_s|^{1/2} dB_s \qquad \text{and} \qquad \int_0^t (B_s + s)^2 ds.$$

6. **Optional bonus problem:** The integrals

$$I_1 = \int_0^t B(s)ds \quad \text{and} \quad I_2 = \int_0^t B(s)^2 ds$$

are *not* stochastic integrals, although they are random variables (and define natural stochastic processes). For each ω the integrands are nice continuous functions of s and the ds integration is just the traditional calculus integration. Find the mean and variance of the the random variables I_1 and I_2.