Statistics 491, Problem Set 10

Wellner; 11/27/13

Reading: Durrett; Chapter 2, pages 92 - 118 .
Ross; Chapter 5, sections 5.1-5.4+Exercises, pages 291 - 370.

Due: Wednesday, December 4, 2013.

1. Durrett, chapter 2, problem 2.2, page 111.
The life time of a radio is exponentially distributed with mean 5 years. If Ted buys a 7 year-old radio, what is the probability it will be working 3 years later?

2. Durrett, chapter 2, problem 2.34, page 115.
Edwin catches trout at times of a poisson process with rate 3 per hour. Suppose that the trout weigh an average of 4 pounds with a standard deviation of 2 pounds. Find the mean and standard deviation of the total weight of fish he catches in 2 hours.

3. Durrett, chapter 2, problem 2.38, page 115.
Let S_t be the price of a stock at time t and suppose that at times of a Poisson process with rate λ the price is multiplied by a random variable $X_i > 0$ with mean μ and variance σ^2. That is,

$$S_t = S_0 \prod_{i=1}^{N(t)} X_i$$

where the product is 1 if $N(t) = 0$. Find $E(S_t)$ and $Var(S_t)$ as functions of λ, t, μ, and σ^2.

4. Durrett, chapter 2, problems 2.57 and 2.58, page 118.
2.57: Suppose $N(t)$ is a Poisson process with rate 2. Compute the conditional probabilities: (a) $P(N(3) = 4|N(1) = 1)$. (b) $P(N(1) = 1|N(3) = 4)$.
2.58: For a Poisson process $N(t)$ with rate 2 compute:
(a) $P(N(2) = 5)$. (b) $P(N(5) = 8|N(2) = 3)$. (c) $P(N(2) = 3|N(5) = 8)$.

5. Durrett, chapter 2, problem 2.32, page 114.
Let T be exponentially distributed with rate λ.
(a) Use the definition of conditional expectation to compute $E(T|T > c)$.
(b) Determine $E(T|T < c)$ from the identity

$$E(T) = E(T|T < c)P(T < c) + E(T|T > c)P(T > c).$$

6. Durrett, chapter 2, problem 2.39: Messages arrive to be transmitted across the internet at ties of a poisson process with rate λ. Let Y_i be the size of the ith message, measured in bytes, and let $G(z) = E(z^{Y_i})$ be the generating function of Y_i (with assumptions as in Example 2.2 page 104). Let $N(t)$ be the number of arrivals by time t and let $S_t = Y_1 + \cdots + Y_{N(t)}$ be the total size of the messages up to time t.
   (a) Find the generating function $f(z) = E(z^{S_t})$.
   (b) Differentiate and set $z = 1$ to find $E(S)$.
   (c) Differentiate again and set $z = 1$ to find $E\{S_t(S_t - 1)\}$.
   (d) Compute $Var(S_t)$.

7. **Optional bonus problem 1:** Durrett, chapter 2, problem 2.32, page 115. (When did the chicken cross the road?) Suppose that traffic on a road follows a Poisson process with rate λ cars per minute. A chicken news a gap of length at least c minutes in the traffic to cross the road. To compute the time the chicken will have to wait to cross the road, let τ_1, τ_2, τ_3, ... be the inter arrival times for the cars and let $J \equiv \min\{j : \tau_j > c\}$. If $T_n = \tau_1 + \cdots + \tau_n$, then the chicken will start to cross the road at time T_{J-1} and complete his (or her) journey at time $T_{J-1} + c$. Use the previous exercise to show that $E(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$.

8. **Optional bonus problem 2:** Durrett, chapter 2, problem 2.62, page 118. Consider two independent Poisson processes $N_1(t)$ and $N_2(t)$ with rates λ_1 and λ_2. What is the probability that the two-dimensional process $(N_1(t), N_2(t))$ ever visits the point (i, j)?

9. **Optional bonus problem 3:** From Serfling (1975), *Annals of Probability* **3**, 726-731. Let $Y_i \sim \text{Poisson}(p_i)$ and let $Z_i \sim \text{Bernoulli}(1 - e^{p_i}(1 - p_i))$ be independent of Y_i. Define

$$X_i = 1_{[Y_i \geq 1]} + 1_{[Y_i = 0]} \cdot 1_{[Z_i = 1]}.$$

   (a) Show that $X_i \sim \text{Bernoulli}(p_i)$.
   (b) Show that $P(X_i \neq Y_i) = P(Y_i \geq 2) + P(X_i = 1, Y_i = 0) = p_i(1 - e^{-p_i}) \leq p_i^2$.
   (c) Now let $S_n \equiv X_1 + \cdots + X_n$ and $T_n \equiv \sum_{i=1}^n Y_i$. Use the computations in (a) and (b) to show that

$$|P(S_n \in A) - P(T_n \in A)| \leq \sum_{i=1}^n p_i^2.$$