Statistics 491, Problem Set 3

Wellner; 10/9/13

Reading: Ross; Chapter 3, pages 140 - 189

Due: Wednesday, October 16, 2013.

1. Ross, problem 80, page 186, Chapter 3.

2. Ross, problem 88, page 187, Chapter 3.

3. Ross, problem 95, page 189, Chapter 3.

4. Ross, problem 94, page 188, Chapter 3.

5. Ross, problem 97, page 189, Chapter 3.

6. Ross, problem 45, page 180, Chapter 3.

7. First here is some notation for the multinomial distribution of problem 17, Ross, page 88, chapter 2. If $\underline{X}_n \equiv \underline{X} = (X_1, \ldots, X_r)$ is the vector of counts of the r different possible outcomes of an experiment on n independent trials, with probability vector $\underline{p} = (p_1, \ldots, p_r)$ with $\sum_{j=1}^r p_j = 1$, then we say that $\underline{X}$ has a Multinomial distribution with r cells, n trials, and probabilities for the cells given by $\underline{p}$, and we write $\underline{X} \sim \text{Mult}_r(n, \underline{p})$. Notice that this means that we can write $\underline{X} = \sum_{i=1}^n \underline{Y}_i$ where $\underline{Y}_i = (Y_{i,1}, \ldots, Y_{i,r})$ are independent and identically distributed $\text{Mult}_r(1, \underline{p})$ random vectors. (This is exactly analogous to $X \sim$ Binomial(n, p) being equal to the sum of n i.i.d. Bernoulli(p) random variables.) Now suppose that $N \sim$ Poisson (λ) is independent of $\underline{Y}_1, \underline{Y}_2, \ldots$, and hence of $\underline{X}_n$ for each n. Find the distribution of the random vector $\underline{X}_N = \sum_{i=1}^N \underline{Y}_i$. This is the multinomial (vector) analogue of problem 6 of problem set 2.

8. Suppose that a process X_n taking values in $[0, 1]$ is defined as follows: for $0 < \alpha < 1$ and $\beta = 1 - \alpha$ define $\{X_n : \ n \geq 0$ by $X_0 = x \in [0, 1]$ and

$$X_{n+1} = \begin{cases} \alpha + \beta X_n & \text{with probability } X_n \\ \beta X_n & \text{with probability } 1 - X_n. \end{cases}$$

   (a) Show that the process $\{X_n : \ n \geq 0\}$ is a martingale.
   (b) Use the result of (a) to compute $E(X_n)$.
   (c) Find $Var(X_{n+1}|X_n)$ and use this to find a recursive formula for $Var(X_{n+1})$.
   (d) What can you say about the behavior of $Var(X_n)$ as n increases?