Statistics 491, Problem Set 4

Wellner; 10/16/13

Reading: Ross; Chapter 4, pages 191 - 230
Durrett; Chapter 1, pages 1-40 .

Due: Wednesday, October 23, 2013.

1. Suppose that $X_1, X_2, \ldots$ be i.i.d. random variables with $P(X_i = 2) = 1/3$ and $P(X_i = 1/2) = 2/3$. Let $M_0 \equiv 1$ and $M_n = \prod_{i=1}^n X_i$ for $n \geq 1$.
(a) Show that M_n is a martingale with respect to $\mathcal{F}_n \equiv \sigma\{M_0, M_1, \ldots, M_n\}$.
(b) Use the weak (or strong) law of large numbers to how that $M_n \rightarrow_p 0$ (or $\rightarrow_{a.s} 0$).
(c) Show that $\sup_{n\geq 1} E(M_n^r) = \infty$ for every $r > 1$. (This gives an example of a martingale which converges almost surely, but which is not closed at infinity: i.e. $\{M_n : \ n \in \{0, 1, \ldots, \infty\}\}$ is *not a martingale*.

2. Suppose that $Y_1, Y_2, \ldots, Y_n$ are i.i.d. Poisson(1), and let $N_n = \sum_{i=1}^n Y_i$. Thus by the central limit theorem
$$Z_n = \frac{N_n - n}{\sqrt{n}} \rightarrow Z \sim N(0, 1);$$
that is $P(Z_n \leq z) \rightarrow P(Z \leq z) = \Phi(z) = \int_{-\infty}^z (2\pi)^{-1/2} e^{-x^2/2} dx$.
(a) Show that
$$E\left(\frac{n - N_n}{\sqrt{n}} 1_{[N_n \leq n]}\right) = \frac{\sqrt{n}}{n!}\left(\frac{n}{e}\right)^n .$$
(b) Argue that $-Z_n \rightarrow -Z \stackrel{d}{=} Z \sim N(0, 1)$, so by a integrability argument
$$E\left(\frac{n - N_n}{\sqrt{n}} 1_{[n - N_n \geq 0]}\right) \rightarrow E(Z 1_{[Z \geq 0]}) = \frac{1}{\sqrt{2\pi}}.$$
(c) Conclude that
$$\frac{\sqrt{2\pi n}(\frac{n}{e})^n}{n!} \rightarrow 1.$$
The resulting approximation, $n! \sim \sqrt{2\pi n}(n/e)^n$ is known as *Stirling's formula*.

3. Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. Poisson(λ) random variables.
(a) Find the moment generating function of the X_i's, $\phi(s) \equiv E \exp(s X_1)$.
(b) Find the form of the exponential martingale $M_n = \exp(s \sum_{j=1}^n X_j)/\phi(s)^n$.

(c) Consider another value of the Poisson parameter, $\mu > 0$ say, and the likelihood ratio $p(x;\mu)/p(x;\lambda)$ where $p(x;\gamma) = e^{-\gamma}\gamma^x/x!$ is the probability mass function for the Poisson(γ) distribution. Show that for some choice of s in (b) the exponential martingale becomes exactly the likelihood ratio martingale

$$Z_n \equiv \prod_{i=1}^{n} \frac{p(X_i;\mu)}{p(X_i;\lambda)}.$$

4. Suppose that $U \sim \text{Uniform}(0,1)$. For $n \geq 1$ let $Z_{n,k} \equiv 1_{(k-1)/2^n, k/2^n]}(U)$ for $k \in \{1, \ldots, 2^n\}$, and write $\underline{Z}_n \equiv (Z_{n,1}, \ldots, Z_{n,2^n})$. Now let f be a function defined on $[0,1]$ with $\int_0^1 f^2(x)dx < \infty$ and define $Y_\infty \equiv f(U)$.
(a) Show that $E(Y^2) < \infty$.
(b) Compute $Y_n \equiv E(Y|\underline{Z}_n) = E(Y|Z_{n,1}, \ldots, Z_{n,2^n})$ explicitly as a function of $\underline{Z}_n$.
(c) Show that $\{Y_n : n \geq 1\}$ is a martingale.
(d) Find a predictable process $A_n = \langle Y \rangle_n$ such that $Y_n^2 - \langle Y \rangle_n$ is a martingale.
(e) Can you show that $E(Y - Y_n)^2 \to 0$?

5. Suppose that X_i are i.i.d as 2Bernoulli(p) $- 1$ (so $P(X_i = 1) = p$ and $P(X_i = -1) = q = 1 - p$). Let $S_n \equiv \sum_{i=1}^n X_i$, and let $T_b \equiv T \equiv \min\{n \geq 1 : S_n = b\}$.
(a) If $p = 1/3$, compute $p_b = P(\max_{n\geq 1} S_n \geq b)$, $E(T_b)$, and $E(\max_{n\geq 1} S_n)$. In particular find p_b and $E(T_b)$ for $b \in \{1,2,3,4\}$.
(b) Repeat the calculations in (a) when $p = 9/19$.

6. Now suppose that the X_i's are as in the previous problem with $p = 1/2$, define $S_n = \sum_{i=1}^n X_i$, and let

$$T = \min\{n \geq 1 : S_n = 1\}.$$

Let $\mathcal{F}_n = \sigma\{S_0, S_1, \ldots, S_n\}$. Then T is a stopping time and by our exponential martingale example 4, with

$$\phi(\theta) = (1/2)e^{\theta} + (1/2)e^{-\theta} = \cosh(\theta),$$

and hence

$$M_n = \exp(\theta S_n)/\phi(\theta)^2 = \frac{e^{\theta S_n}}{\cosh(\theta)^n} = (\text{sech}\theta)^n e^{\theta S_n}$$

is a martingale with respect to $\mathcal{F}_n$: $E(M_{n+1}|\mathcal{F}_n) = M_n$.
(a) Use the optional sampling theorem applied to the martingale $\{M_n\}$ and the bounded stopping time $T \wedge k$ to find an identity for $EM_{T\wedge k}$.
(b) Use the result of (a) to find an identity for EM_T and hence an expression for $E\{(\text{sech}\theta))^T\}$ when $\theta > 0$

(c) Show that $(\text{sech}\theta)^T \nearrow 1$ as $\theta \searrow 0$ if $T < \infty$, and that $(\text{sech}\theta)^T \nearrow 0$ as $\theta \searrow 0$ if $T = \infty$. Use these to show that $P(T < \infty) = E1_{[T<\infty]} = 1$.
(d) Take $\alpha = \text{sech}(\theta)$ in the identity in (b); then show that on the one hand

$$E(\alpha^T) = \sum_{n=1}^{\infty} \alpha^n P(T = n) = e^{-\theta} \tag{1}$$

while on the other hand $g(\alpha) \equiv E(\alpha^T)$ satisfies

$$g(\alpha) = \frac{1}{2}\alpha + \frac{1}{2}\alpha g(\alpha)^2 \tag{2}$$

and hence $g(\alpha) = E(\alpha^T) = \alpha^{-1}(1 - \sqrt{1-\alpha^2})$. (Show this by conditioning on X_1.)
(e) Use the result of (d) to show that $P(T = 2m-1) = (-1)^{m+1}\binom{1/2}{m}$ for $m \geq 1$ where for $x \in \mathbb{R}$ and $r \in \mathbb{N}$

$$\binom{x}{r} \equiv \frac{x(x-1)\cdots(x-r+1)}{r!}.$$

(f) Use the result in (e) to show that $E(T) = \infty$.