Arcsin Laws and Feynman-Kac

Math / Statistics 523

Wellner; 26 May 2017, 27 May 2020

1 Two arcsin laws for Brownian motion

Let $\{\mathbb{B}(t) : \; t \ge 0\}$ denote standard Brownian motion on $[0, \infty)$. Consider the following random variables:

$$
\begin{aligned}
L &\equiv \sup\{t \in [0,1] : \; \mathbb{B}(t) = 0\}, \\
M^* &\equiv \text{argmax}_{0 \le t \le 1} \mathbb{B}(t), \\
T^+ &\equiv \lambda(\{t \in [0,1] : \; \mathbb{B}(t) > 0\}) = \int_0^1 1_{(0,\infty)}(\mathbb{B}(s)) ds.
\end{aligned}
$$

The two arcsin laws for Brownian motion are as follows:

Theorem 1. (First arcsin law) Both L and M^* have the arcsin distribution on $[0, 1]$. That is,

$$
\begin{aligned}
P(L \le u) &= \frac{2}{\pi} \arcsin(\sqrt{u}) \;\; \text{for} \;\; 0 \le u \le 1\}, \;\; \text{and} \\
P(M^* \le u) &= \frac{2}{\pi} \arcsin(\sqrt{u}) \;\; \text{for} \;\; 0 \le u \le 1\}.
\end{aligned}
$$

The corresponding densities are $f_L(u) = f_{M^*}(u) = 1/(\pi\sqrt{u(1-u)}) 1_{(0,1)}(u)$. (This is exactly the Beta$(1/2, 1/2)$ density.)

Theorem 2. (Second arcsin law) T^+ has the arcsin distribution on $[0, 1]$.

Remark. Note that if $T_t^+ \equiv \int_0^t 1_{(0,\infty)}(\mathbb{B}_s) ds$, then

$$
t^{-1} T_t^+ \stackrel{d}{=} T_1^+ \equiv T^+ = \int_0^1 1_{(0,\infty)}(\mathbb{B}_u) du.
$$

Proof. The proof of the arcsin law for L is straightforward:

$$\begin{aligned}
P(L \le s) &= \int_{-\infty}^{\infty} p_s(0,y) P_y(\tau_0 > 1-s) dy \\
&= 2\int_0^{\infty} \frac{1}{\sqrt{s}} \phi\left(\frac{y}{\sqrt{s}}\right) \left(\int_{1-s}^{\infty} \frac{y}{\sqrt{2\pi v^3}} \exp\left(-\frac{y^2}{2v}\right) dv \right) dy \\
&= \frac{1}{\pi} \int_{1-s}^{\infty} \frac{1}{\sqrt{sv^3}} \int_0^{\infty} y e^{-y^2(v+s)/2vs} dy dv \\
&= \frac{1}{\pi} \int_{1-s}^{\infty} \frac{1}{\sqrt{sv^3}} \frac{vs}{v+s} dv = \frac{1}{\pi} \int_{1-s}^{\infty} \left(\frac{(v+s)^2}{vs} \right)^{1/2} \frac{s}{(v+s)^2} dv \\
&= \frac{1}{\pi} \int_0^s \frac{1}{\sqrt{t(1-t)}} dt \quad \text{by letting } \ t = s/(v+s) \\
&= \frac{2}{\pi} \arcsin(\sqrt{s}).
\end{aligned}$$

This argument followed Durrett (2010).
The proof for M^* relies on two facts:
Fact 1: M^* is uniquely defined.
Fact 2: (Lévy) Let $\mathbb{M}(t) \equiv \sup_{s \le t} \mathbb{B}(s)$. Then $\mathbb{Y}(t) \equiv \mathbb{M}(t) - \mathbb{B}(t)$ is a reflected Brownian motion; i.e. $\mathbb{Y}(t) \stackrel{d}{=} |\mathbb{B}(t)|$.

Now the arcsin law holds for M^* by recognizing that M^* is the last zero of $\mathbb{Y} = \mathbb{M} - \mathbb{B}$ before time 1. But here is a separate proof. Note that

$$\begin{aligned}
P(M^* < s) &= P(\sup_{0 \le v \le s} \mathbb{B}(v) > \sup_{s \le v \le 1} \mathbb{B}(v)) \\
&= P(\sup_{0 \le v \le s} \mathbb{B}(v) - \mathbb{B}(s) > \sup_{s \le v \le 1} \mathbb{B}(v) - \mathbb{B}(s)) \\
&= P(\mathbb{Y}_1(s) > \mathbb{Y}_2(1-s)) \\
&= P(|\mathbb{B}_1(s)| > |\mathbb{B}_2(1-s)|)
\end{aligned}$$

where $\{\mathbb{M}_1(t) : \ 0 \le t \le s\}$ is the maximum process of $\mathbb{B}_1(t) \equiv \mathbb{B}(s-t) - \mathbb{B}(s)$, $0 \le t \le s$, and $\{\mathbb{M}_2(t) : \ 0 \le t \le 1-s\}$ is the maximum process of $\mathbb{B}_2(t) \equiv \mathbb{B}(s+t) - \mathbb{B}(s)$, $0 \le t \le 1-s$. But then, since $\mathbb{Y}_1 \stackrel{d}{=} |\mathbb{B}_1|$ and $\mathbb{Y}_2 \stackrel{d}{=} |\mathbb{B}_2|$, the last line in the last display equals, with Z_1 and Z_2 denoting independent standard normal random variables,

$$\begin{aligned} P(|\mathbb{B}_1(s)| > |\mathbb{B}_2(1-s)|) &= P(\sqrt{s}|Z_1| > \sqrt{1-s}|Z_2|) \\ &= P\left(\frac{|Z_2|}{\sqrt{Z_1^2+Z_2^2}} < \sqrt{s}\right) \\ &= P(|\sin(\Theta)| < \sqrt{s}) \quad \text{where } \; \Theta \sim \text{Uniform on } [0, 2\pi] \\ &= 4P(\Theta < \arcsin(\sqrt{s})) = \frac{4}{2\pi}\arcsin(\sqrt{s}) \\ &= \frac{2}{\pi}\arcsin(\sqrt{s}). \end{aligned}$$

□

This proof of the arcsin law for M^* followed Mörters and Peres (2010), pages 136-137.

2 Feynman - Kac for Brownian motion

To prove the second arcsin law we will first introduce some tools linking additive functionals of Brownian motion to differential equations.

Let $U \subset \mathbb{R}^d$ be open and bounded, or let $U = \mathbb{R}^d$. A function $u : [0, \infty) \times U \to [0, \infty)$ solves the heat equation with dissipation rate $V : U \to \mathbb{R}$ and initial condition f if:
(i) $\lim_{x \to x_0, t \searrow 0} u(t, x) = f(x_0)$ for all $x_0 \in U$.
(ii) $\lim_{x \to x_0, t \searrow 0} u(t, x) = 0$ whenever $x_0 \in \partial U$.
(iii) u satisfies the following "heat equation with dissipation":

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta_x u(t,x) + V(x)u(t,x) \quad \text{on } (0,\infty)\times U \tag{1}$$

where Δ_x acts on x and is given by

$$\Delta_x \equiv \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \equiv \sum_{j=1}^{d} \partial_{jj}.$$

Theorem. (Feynman - Kac)
(i) Suppose that $V : \mathbb{R}^d \to \mathbb{R}$ is bounded. Then $u : [0, \infty) \times U \to \mathbb{R}$ defined by

$$u(t,x) = E_x\left\{\exp\left(\int_0^t V(\mathbb{B}_s)ds\right)\right\}$$

satisfies the heat equation on U with dissipation rate V and initial condition 1.
(ii) Furthermore, if $\{\mathbb{B}_t : \ t \geq 0\}$ denotes standard Brownian motion,

$$u(t,x) = E_0 \left\{ f(x+\mathbb{B}_t) \exp\left(\int_0^t V(x+\mathbb{B}_s) ds \right) \right\}$$

satisfies (1).

Example 1. When $V \equiv 0$, then the ordinary heat equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2} \Delta_x u(t,x)$$

subject to $u(0,x) = f(x)$ has a unique bounded solution given by

$$u(t,x) = E\left\{ f(x+\mathbb{B}_t) \right\} = \int_{\mathbb{R}} f(z) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(z-x)^2}{2t} \right) dz.$$

Example 2. For a non-negative random variable T independent of $\mathbb{B}$, define a new process $\{X_t : \ t \geq 0\}$ by

$$X_t \equiv \begin{cases} \mathbb{B}_t, & 0 \leq t \leq T, \\ \Delta = \text{"coffin"}, & \text{if } T < t. \end{cases}$$

For $f : \mathbb{R} \to \mathbb{R}$, define $f : \mathbb{R} \cup \{\Delta\}$ by $f(\Delta) = 0$. Now take $T \sim$ exponential(λ). Then X is exponentially killed BM with killing rate λ. Does

$$u(t,x) \equiv E\{f(x+X_t)\}$$

satisfy some interesting equation? Now $u(0,x) = f(x)$ since $X_0 = 0$. Furthermore,

$$u(t,x) = E\{f(x+\mathbb{B}_t) 1_{[T>t]}\} = e^{-\lambda t} E\{f(x+\mathbb{B}_t)\}$$

by independence of T and $\mathbb{B}$. Hence we find that

$$\begin{aligned}
\frac{\partial}{\partial t} u(t,x) &= e^{-\lambda t} \frac{\partial}{\partial t} E\{f(x+\mathbb{B}_t)\} - \lambda e^{-\lambda t} E\{f(x+\mathbb{B}_t)\} \\
&= e^{-\lambda t} \frac{1}{2} \frac{\partial^2}{\partial x^2} E\{f(x+\mathbb{B}_t)\} - \lambda E\{f(x+\mathbb{B}_t) 1_{[T>t]}\} \\
&= \frac{1}{2} \frac{\partial^2}{\partial x^2} E\{f(x+\mathbb{B}_t) 1_{[T>t]}\} - \lambda E\{f(x+\mathbb{B}_t) 1_{[T>t]}\} \\
&= \frac{1}{2} \frac{\partial^2}{\partial x^2} u(t,x) - \lambda u(t,x).
\end{aligned}$$

so Feynman - Kac holds when $V(x) \equiv -\lambda$. (These examples are from Steele (2001).)

There are several possible proofs of the Feynman - Kac theorem. The most direct proof (and the route taken by Kac (1951)) involves expansion of the exponential, direct calculation, and summing back up; see e.g. Mörters and Peres (2010), pages 213-215. The proof we sketch here is from Steele (2001). It involves construction of an interpolating martingale.

Proof. We want to find a martingale $\{M_s : \ 0 \le s \le t\}$ such that $M_0 = u(t,x)$ and

$$E\{M_t\} = E_0 \left\{ f(x+\mathbb{B}_t) \exp\left(\int_0^t V(x+\mathbb{B}_s) ds \right) \right\}.$$

To this end, define $\{M_s : 0 \le s \le t\}$ by

$$M_s \equiv u(t-s, x+\mathbb{B}_s) \exp\left(\int_0^s V(x+\mathbb{B}_v) dv \right).$$

First note that $M_0 = u(t,x)$. By Ito's formula and the assumption that $u(t,x)$ solves the heat equation (1) we have

$$\begin{aligned} du(t-s, x+\mathbb{B}_s) &= u_x(t-s, x+\mathbb{B}_s) d\mathbb{B}_s + \frac{1}{2} u_{xx}(t-s, x+\mathbb{B}_s) ds - u_t(t-s), x+\mathbb{B}_s) ds \\ &= u_x(t-s, x+\mathbb{B}_s) d\mathbb{B}_s - V(x+\mathbb{B}_s) ds. \end{aligned}$$

Then, by the product rule,

$$\begin{aligned} dM_s &= \exp\left(\int_0^s V(x+\mathbb{B}_v) dv \right) \{ u_x(t-s, x+\mathbb{B}_s) d\mathbb{B}_s - V(x+\mathbb{B}_s) u(t-s, x+\mathbb{B}_s) ds \} \\ &\quad + \exp\left(\int_0^s V(x+\mathbb{B}_v) dv \right) V(x+\mathbb{B}_s) u(t-s, x+\mathbb{B}_s) ds \\ &= \exp\left(\int_0^s V(x+\mathbb{B}_v) dv \right) u_x(t-s, x+\mathbb{B}_s) d\mathbb{B}_s. \end{aligned}$$

It follows that $\{M_s : \ 0 \le s \le t\}$ is a (local-) martingale. In fact, from the definition of M it follows that

$$\sup_{0 \le s \le t} |M_s| \le \|u\|_\infty \exp(t \|V\|_\infty),$$

and this is enough ensure that $\{M_s:\ 0 \le s \le t\}$ is an honest martingale. This yields the desired conclusion

$$u(t,x) = M_0 = E(M_t) = E\left\{ f(x+\mathbb{B}_t)\exp\left(\int_0^t V(x+\mathbb{B}_s)ds\right)\right\}.$$

3 Proof of the second arcsin law via Feynman - Kac

To use the Feynman - Kac theorem to prove that the second arcsin law holds, let

$$V(x) = -\lambda 1_{(0,\infty)}(x) \quad \text{and} \quad f(x) = 1.$$

Then

$$u(t,x) \equiv E_x \exp\left(-\lambda \int_0^t 1_{(0,\infty)}(\mathbb{B}(s))ds\right)$$

satisfies

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}u_{xx}(t,s) - \lambda 1_{(0,\infty)}(x)u(t,x)$$

subject to $u(0,x) = 1$ for all $x \in \mathbb{R}$. This can be written as

$$u_t(t,x) = \begin{cases} \frac{1}{2}u_{xx}(t,x) - \lambda u(t,x), & x > 0, \\ \frac{1}{2}u_{xx}(t,x), & x \le 0. \end{cases} \tag{1}$$

Now let

$$\hat{u}(\alpha,x) \equiv \int_0^\infty e^{-\alpha t}u(t,x)dt,$$

the Laplace transform (on t) of $u(t,x)$. Then the Laplace transform of $u_t(t,x)$ becomes $-1+\alpha\hat{u}(\alpha,x)$, and the Laplace transform (on t) of $u_{xx}(t,x)$ is just $\hat{u}_{xx}(\alpha,x)$. Then the system of equations (1) becomes

$$-1+\alpha\hat{u}(\alpha,x) = \begin{cases} \frac{1}{2}\hat{u}_{xx}(\alpha,x) - \lambda\hat{u}(\alpha,x), & x > 0, \\ \frac{1}{2}\hat{u}_{xx}(\alpha,x), & x \le 0; \end{cases}$$

or, alternatively, the system of ordinary differential equations

$$\begin{aligned} \frac{1}{2}\hat{u}_{xx}(\alpha,x) - (\alpha+\lambda)\hat{u}(\alpha,x) &= -1, \quad x > 0, \\ \frac{1}{2}\hat{u}_{xx}(\alpha,x) - \alpha\hat{u}(\alpha,x) &= -1, \quad x \le 0. \end{aligned}$$

Using standard methods from ODE theory we find that the only bounded solutions are given by

$$\hat{u}(\alpha, x) = \begin{cases} \frac{1}{\alpha+\lambda} + c_0 \exp(-x\sqrt{2(\alpha+\lambda)}), & x > 0, \\ \frac{1}{\alpha} + c_1 \exp(x\sqrt{2\alpha}), & x \le 0. \end{cases}$$

In order for $u \in C^2$, we must have

$$u(t, 0^+) = u(t, 0^-) \quad \text{and} \quad u_x(t, 0^+) = u_x(t, 0^-),$$

and hence

$$\hat{u}(\alpha, 0^+) = \hat{u}(\alpha, 0^-) \quad \text{and} \quad \hat{u}_x(\alpha, 0^+) = \hat{u}_x(\alpha, 0^-).$$

The first equation yields

$$\frac{1}{\alpha+\lambda} + c_0 = \frac{1}{\alpha} + c_1,$$

and the second equation gives

$$-c_0\sqrt{2(\alpha+\lambda)} = c_1\sqrt{2\alpha}.$$

Solving these gives

$$\begin{aligned} c_0 &= \frac{\sqrt{\alpha+\lambda} - \sqrt{\alpha}}{\sqrt{\alpha}(\alpha+\lambda)}, \\ c_1 &= \frac{\sqrt{\alpha} - \sqrt{\alpha+\lambda}}{\sqrt{\alpha}(\sqrt{\alpha+\lambda})}. \end{aligned}$$

Hence we find that

$$\begin{aligned} \hat{u}(\alpha, 0^+) &= \int_0^\infty e^{-\alpha t} u(t, 0) dt = \frac{1}{\alpha+\lambda} + c_0 \\ &= \frac{1}{\alpha+\lambda}\left\{1 + \frac{\sqrt{\alpha+\lambda} - \sqrt{\alpha}}{\sqrt{\alpha}}\right\} \\ &= \frac{1}{\sqrt{\alpha(\alpha+\lambda)}}. \end{aligned}$$

This is, in fact, a closely related to a Stieltjes transform. We can either invert it directly (see e.g. Widder (1941), or the derivation in a slightly more complicated problem in Keilson and Wellner (1978)), or proceed backward from the conjectured result. From the equality in distribution noted after the statement of the second arcsin law we want to show that

$$E\exp(-\lambda T_t^+) = \int_0^t \frac{1}{\pi}\frac{e^{-\lambda s}}{\sqrt{s(t-s)}}ds.$$

Computing the Laplace transform on t, we equivalently want to show that

$$\begin{aligned}
\frac{1}{\sqrt{\alpha(\alpha+\lambda)}} &= \int_0^\infty e^{-\alpha t}E(e^{-\lambda T_t^+})dt \\
&\stackrel{?}{=} \int_0^\infty \int_0^t e^{-\alpha t}\frac{1}{\pi}\frac{e^{-\lambda s}}{\sqrt{s(t-s)}}dsdt.
\end{aligned}$$

But the double integral on the right side, equals, via Fubini's theorem,

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{s}}\left(\int_s^\infty \frac{e^{-\alpha t}}{\sqrt{t-s}}dt\right)ds \\
&= \frac{1}{\pi}\int_0^\infty \frac{e^{-\lambda s}}{\sqrt{s}}\left(\int_0^\infty \frac{e^{-\alpha(t+s)}}{\sqrt{t}}dt\right)ds \\
&= \frac{1}{\pi}\int_0^\infty \frac{e^{-(\alpha+\lambda)s}}{\sqrt{s}}\left(\int_0^\infty \frac{e^{-\alpha t}}{\sqrt{t}}dt\right)ds \\
&= \frac{1}{\pi}\sqrt{\frac{\pi}{\alpha+\lambda}}\sqrt{\frac{\pi}{\alpha}} \\
&= \frac{1}{\sqrt{\alpha(\alpha+\lambda)}}
\end{aligned}$$

as claimed. This completes the proof of the second arcsin law. □

4 A Generalized Feynman - Kac Formula

The following theorem is from section 15.4 of Steele (2001). For further related results see Karatzas and Shreve (1991), pages 268-271 and Durrett (1984), chapter 8.

Theorem. (Generalized Feynman - Kac). Suppose that $V : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are bounded and suppose that $u(t, x)$ is the solution of the partial differential equation

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\sigma^2(x) u_{xx}(t,x) + \mu(x) u_x(t,x) + V(x) u(t,x)$$

subject to the initial condition $u(0, x) = f(x)$. If $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz and satisfy $\mu^2(x) + \sigma^2(x) \leq A(1 + x^2)$ for some $A > 0$, then

$$u(t,x) = E\left\{ f(x + X_t) \exp\left(\int_0^t V(x + X_s) ds \right) \right\}$$

where the process X_t is the unique solution of the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)d\mathbb{B}_t, \qquad X_0 = 0.$$

Note that μ and σ are assumed to satisfy a Lipschitz condition. In the following we are interested in the simple (non-Lipschitz) case of

$$\sigma^2(x) = \sigma_+^2 1_{[0,\infty)}(x) + \sigma_-^2 1_{(-\infty,0)}(x)$$

where $\sigma_+^2 \neq \sigma_-^2$. Here we give statements of several results for the process $\mathbb{Y}$ defined as follows: for $x \in \mathbb{R}$ and $\mathbb{B}$ denoting a standard Brownian motion, let $A_x(t)$ be defined by

$$A_x(t) = \int_0^t \sigma^{-2}(x + \mathbb{B}(s)) ds = \int_{\mathbb{R}} L(t, y - x) m(dy)$$

where $m(dy) = 2\sigma^{-2}(y)dy$ and L is local time for $\mathbb{B}$. Then define

$$Y(t) \equiv \mathbb{B}(A_x^{-1}(t)) + x, \quad \text{for} \quad t \geq 0.$$

Keilson & W, Keilson and Wellner (1978) , call Y *oscillating Brownian motion.* It is an inhomogeneous (in space) Brownian motion process with different variance parameters above and below 0. As is easily seen in our particular case

$$A_x(t) = \sigma_-^{-2} t + (\sigma_+^{-2} - \sigma_-^{-2}) \lambda(\{s \leq t : \ \mathbb{B}(s) + x \geq 0\}), \quad t \geq 0.$$

Notation:
(i) The transition density of Y is $p(t, x, y)$; thus

$$P_x(Y(t) \in dy) = p(t,x,y) dm(y) \quad \text{for} \quad \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}.$$

The transition density for standard Brownian motion $\mathbb{B}$ is

$$p^*(t,x,y) = (2\pi t)^{-1/2}\exp(-(y-x)^2/(2t)).$$

Theorem 1. (transition densities for Y). Let $\theta \equiv \sigma_-/(\sigma_+ + \sigma_-) \equiv (1+r)^{-1}$, $r \equiv \sigma_+/\sigma_-$. Then

$$p(t,x,y) = \begin{cases} \frac{1}{2}\theta\sigma_+ \left\{(1+r)p^*\left(t, \frac{x}{\sigma_+}, \frac{y}{\sigma_+}\right) + (1-r)p^*(\left(t, \frac{-x}{\sigma_+}, \frac{y}{\sigma_+}\right)\right\}, & 0 \le y \le x, \\ \theta\sigma_+ p^*(\left(t, \frac{x}{\sigma_+}, \frac{y}{\sigma_-}\right), & y \le 0 \le x, \\ \frac{1}{2}\theta\sigma_+ \left\{(1+r^{-1})p^*\left(t, \frac{x}{\sigma_-}, \frac{y}{\sigma_-}\right) + (1-r^{-1})p^*(\left(t, \frac{-x}{\sigma_-}, \frac{y}{\sigma_-}\right)\right\}, & y \le x \le 0. \end{cases}$$

Here is an arcsin theorem for Y: Let $M(t) \equiv \lambda(\{s \le t : \ Y(s) \ge 0\})$.

Theorem 3. (Occupation time of $\mathbb{R}^+$). For $t > 0$ $M(1) \stackrel{d}{=} M(t)/t$. Furthermore, for $0 < r < \infty$, $M(1)$ has the distribution

$$P_0(M(1) \in du) = \frac{1}{\pi}\frac{1}{\sqrt{u(1-u)}} \cdot \frac{r}{1-(1-r^2)u}du, \quad 0 < u < 1.$$

We also compute $E_0M(1) = \theta$, $Var_0(M(1)) = (1/2)\theta(1-\theta)$.

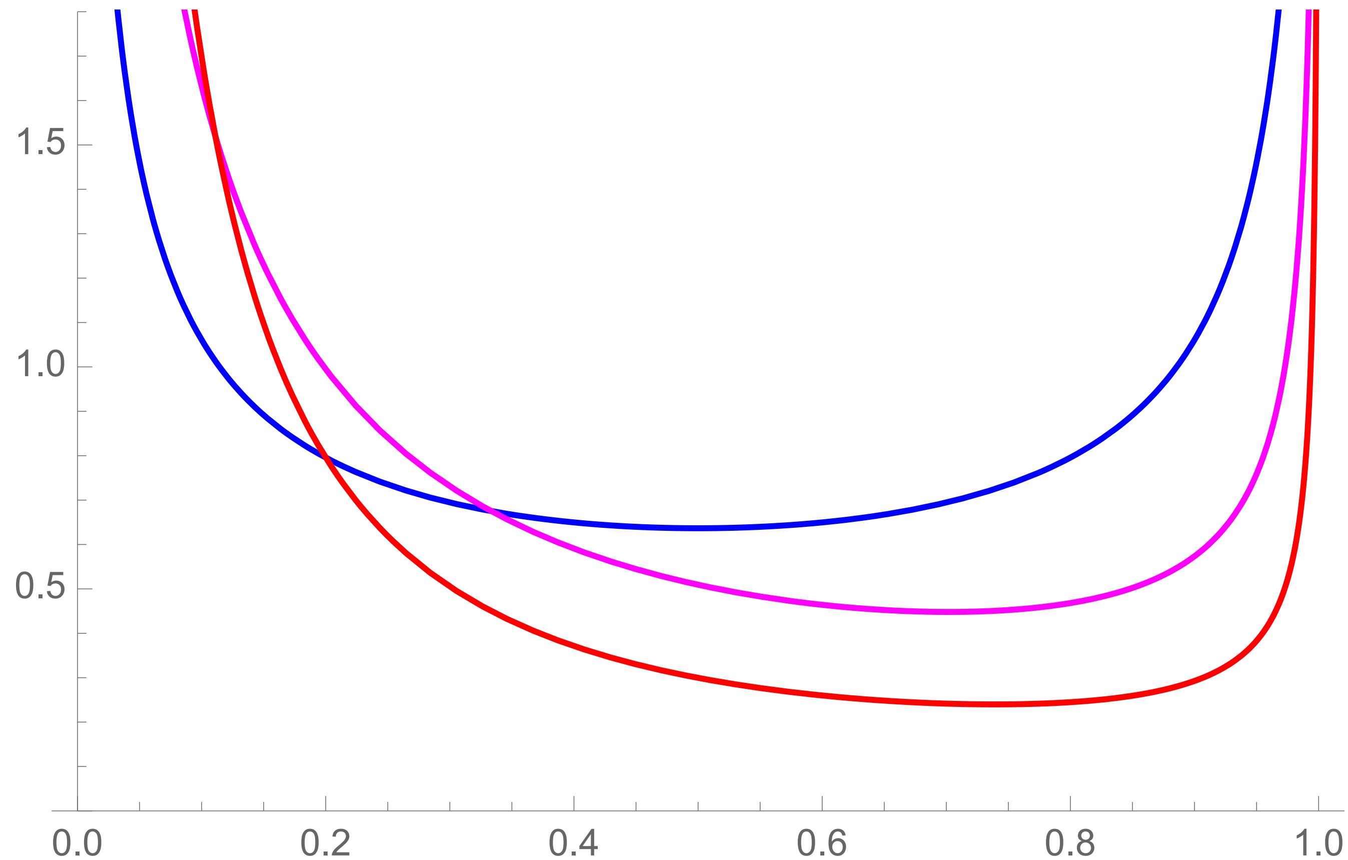
Note that when $\sigma_+ = \sigma_-$ we have $r = 1$ and Theorem 2 reduces to the arcsin law for standard Brownian motion.

Theorem 4. (Occupation time of $\mathbb{R}^+$ conditional on $Y(1) = 0$) For $0 < r < \infty$ and $0 < u < 1$

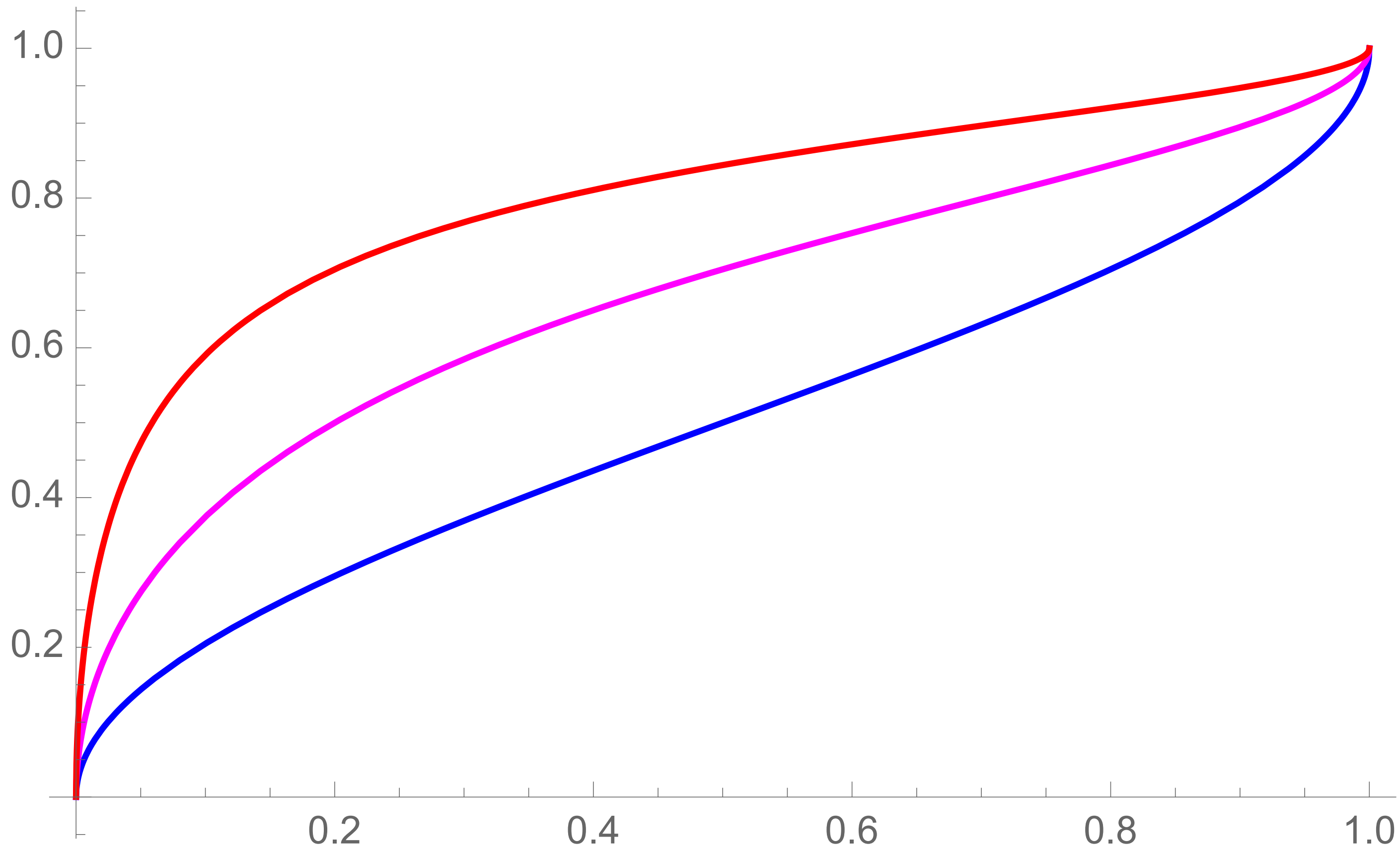
$$P_0(M(1) \le u | Y(1) = 0) = \frac{r(r+1)}{r(u)(r(u)+1)} \cdot u$$

where $r(u) \equiv [1 - u(1-r^2)]^{1/2}$.

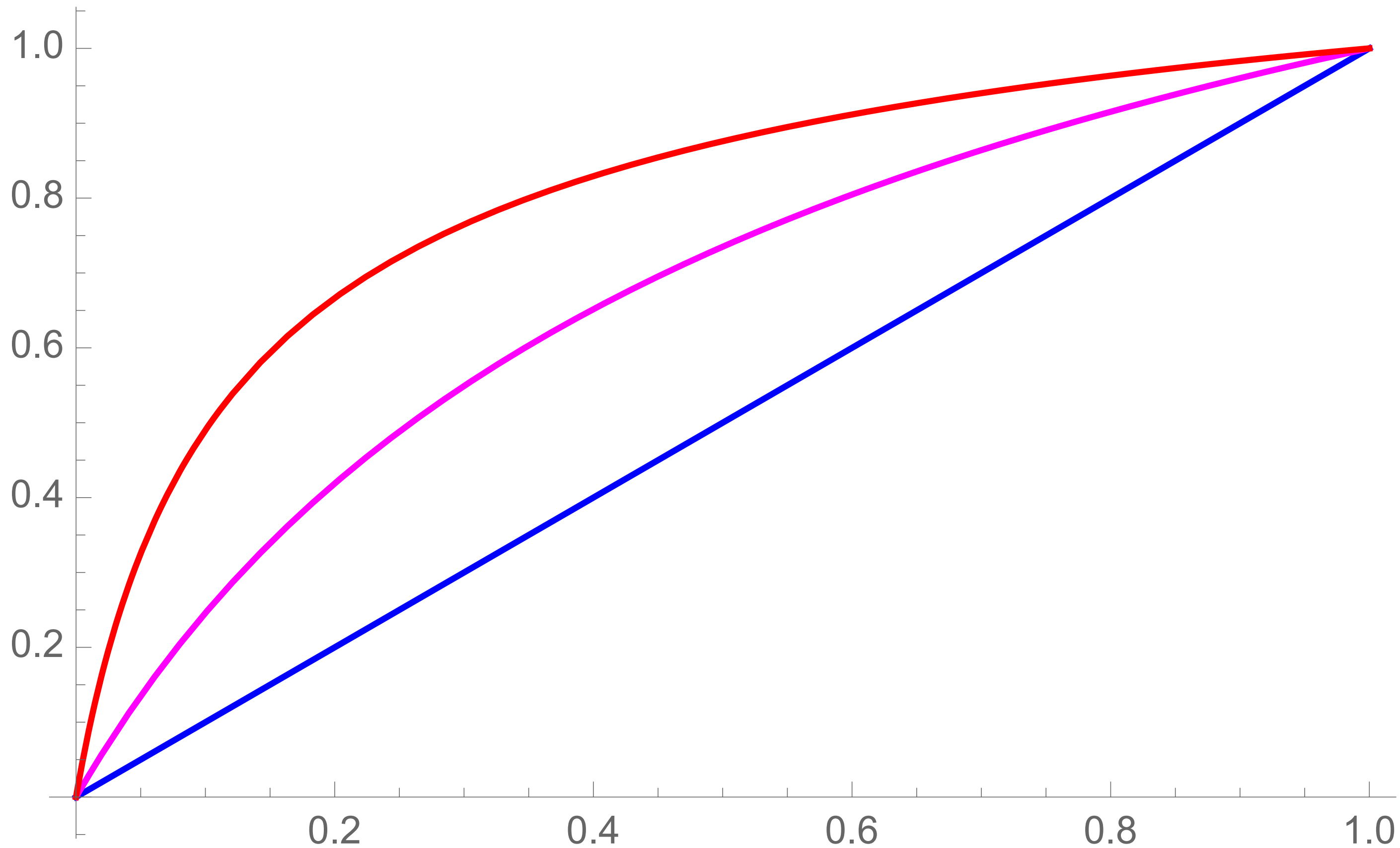
Note that when $\sigma_+ = \sigma_-$ so that $r = 1$ Theorem 4 reduces to the uniform distribution for M which is known for the Brownian bridge process $\mathbb{U} \stackrel{d}{=} (\mathbb{B}|\mathbb{B}(1) = 0)$.



Oscillating BM arcsin densities: $r = 1$ (blue); $r = 2$ (magenta), $r = 4$ (red)



Oscillating BM arcsin distribution functions: $r = 1$ (blue); $r = 2$ (magenta), $r = 4$ (red)



Oscillating Brownian bridge arcsin distribution functions:
$r = 1$ (blue); $r = 2$ (magenta), $r = 4$ (red)

Acknowledgements: The material in this handout has been drawn from the books Durrett (2010), Mörters and Peres (2010), Steele (2001), Itô and McKean (1974), and Karatzas and Shreve (1991), and from the papers Kac (1951), Harrison and Shepp (1981), and Keilson and Wellner (1978), and Wellner (2016). Also note Kesten (1986, 1987).

For further reading on connections between analysis and probability see Durrett (1984) and Bass (1995).

References

Bass, R. F. (1995). *Probabilistic techniques in analysis*. Probability and its Applications (New York), Springer-Verlag, New York.

Durrett, R. (1984). *Brownian motion and martingales in analysis.* Wadsworth Mathematics Series, Wadsworth International Group, Belmont, CA.

Durrett, R. (2010). *Probability: theory and examples*, vol. 31 of *Cambridge Series in Statistical and Probabilistic Mathematics.* 4th ed. Cambridge University Press, Cambridge.

Harrison, J. M. and Shepp, L. A. (1981). On skew Brownian motion. *Ann. Probab.* **9** 309–313.

Itô, K. and McKean, H. P., Jr. (1974). *Diffusion processes and their sample paths.* Springer-Verlag, Berlin-New York. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.

Kac, M. (1951). On some connections between probability theory and differential and integral equations. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950.* University of California Press, Berkeley and Los Angeles.

Karatzas, I. and Shreve, S. E. (1991). *Brownian motion and stochastic calculus*, vol. 113 of *Graduate Texts in Mathematics.* 2nd ed. Springer-Verlag, New York.

Keilson, J. and Wellner, J. A. (1978). Oscillating Brownian motion. *J. Appl. Probability* **15** 300–310.

Kesten, H. (1986). The influence of Mark Kac on probability theory. *Ann. Probab.* **14** 1103–1128.

Kesten, H. (1987). Correction: "The influence of Mark Kac on probability theory" [Ann. Probab. **14** (1986), no. 4, 1103–1128; MR0866337 (88j:60002)]. *Ann. Probab.* **15** 1228.

Mörters, P. and Peres, Y. (2010). *Brownian motion*, vol. 30 of *Cambridge Series in Statistical and Probabilistic Mathematics.* Cambridge University Press, Cambridge. With an appendix by Oded Schramm and Wendelin Werner.

Steele, J. M. (2001). *Stochastic calculus and financial applications*, vol. 45 of *Applications of Mathematics (New York).* Springer-Verlag, New York.

Wellner, J. A. (2016). Fechner's distribution and connections to skew Brownian motion. In *High dimensional probability VII*, vol. 71 of *Progr. Probab.* Springer, [Cham], 203–216.


MARTINGALE DR 100
MARTINGALE CT 44450
STOP