7 Strong Markov Property for Sums of IID RVs

Let $X_1, X_2, \ldots$ be iid and let $S_n \equiv X_1 + \cdots + X_n$. Let $S \equiv (S_1, S_2, \ldots)$.

**Definition 7.1** The integer valued rv $N$ is a stopping time for the sequence of rvs $S_1, S_2, \ldots$ if $[N = k] \in \mathcal{F}(S_1, \ldots, S_k)$ for all $k \geq 1$. It is elementary that

1. $\mathcal{F}_N \equiv \mathcal{F}(S_k : k \leq N)$
2. $\equiv \{ A \in \mathcal{F}(S) : A \cap [N = k] \in \mathcal{F}(S_1, \ldots, S_k) \text{ for all } k \geq 1 \} = (a \sigma$-field),

since it is clearly closed under complements and countable intersections. (Clearly, $[N = k]$ can be replaced by $[N \leq k]$ in the definition of $\mathcal{F}_N$ in (2).)

**Proposition 7.1** Both $N$ and $S_N$ are $\mathcal{F}_N$-measurable.

**Proof.** Now, to show that $[N \leq m] \in \mathcal{F}_N$ we consider $[N \leq m] \cap [N = k]$ equals $[N = k]$ or $\emptyset$, both of which are in $\mathcal{F}(S)$; this implies $[N \leq m] \in \mathcal{F}_N$. Likewise,

(a) $[S_N \leq x] \cap [N = k] = [S_k \leq x] \cap [N = k] \in \mathcal{F}(S_1, \ldots, S_k)$,

implying that $[S_N \leq x] \in \mathcal{F}_N$. \qed

**Theorem 7.1 (The strong Markov property)** If $N$ is a stopping time, then the increments continuing from the random time

3. $\tilde{S}_k \equiv S_{N+k} - S_N$, $k \geq 1$,

have the same distribution on $(R_\infty, B_\infty)$ as does $S_k, k \geq 1$. Moreover, defining $\tilde{S} \equiv (\tilde{S}_1, \tilde{S}_2, \ldots)$,

4. $\mathcal{F}(\tilde{S}) \equiv \mathcal{F}(\tilde{S}_1, \tilde{S}_2, \ldots)$ is independent of $\mathcal{F}_N$ (hence of $N$ and $S_N$).

**Proof.** Let $B \in B_\infty$ and $A \in \mathcal{F}_N$. Now,

(a) $P(\{\tilde{S} \in B\} \cap A) = \sum_{n=1}^{\infty} P(\{\tilde{S} \in B\} \cap A \cap [N = n])$

$= \sum_{n=1}^{\infty} P(\{(S_{n+1} - S_n, S_{n+2} - S_n, \ldots) \in B\} \cap (A \cap [N = n]))$

with $A \cap [N = n] \in \mathcal{F}(S_1, \ldots, S_n)$

$= \sum_{n=1}^{\infty} P(\{(S_{n+1} - S_n, S_{n+2} - S_n, \ldots) \in B\})P(A \cap [N = n])$

$= P(S \in B)\sum_{n=1}^{\infty} P(A \cap [N = n])$

(b) $= P(S \in B)P(A)$.

Set $A = \Omega$ in (b) to conclude that $\tilde{S} \cong S$. Then use $P(\tilde{S} \in B) = P(S \in B)$ to rewrite (b) as

(c) $P(\{\tilde{S} \in B\} \cap A) = P(\tilde{S} \in B)P(A)$,

which is the statement of independence. \qed

**Exercise 7.1** (Manipulating stopping times) Let $N_1$ and $N_2$ denote stopping times relative to an $\mathcal{F}$ sequence of $\sigma$-fields $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$. Show that $N_1 \wedge N_2, N_1 \vee N_2, N_1 + N_2$, and $N_o \equiv i$ are all stopping times.
Definition 7.2 Define waiting times for return to the origin by

\( W_1 \equiv \min\{n : S_n = 0\} \) with \( W_1 = +\infty \) if the set is empty,

\[
\begin{align*}
W_2 &\equiv \min\{n > W_1 : S_n = 0\} \quad \text{with } W_2 = +\infty \text{ if the set is empty}, \\
W_k &\equiv \min\{n > W_{k-1} : S_n = 0\} \quad \text{with } W_k = +\infty \text{ if the set is empty}.
\end{align*}
\]

Then define \( T_k \equiv W_k - W_{k-1} \), with \( W_0 \equiv 0 \), to be the interarrival times for return to the origin.

Proposition 7.2 If \( P(S_n = 0 \text{ i.o.}) = 1 \), then \( T_1, T_2, \ldots \) are well-defined rvs and are, in fact, iid.

Proof. Clearly, each \( W_k \) is always an extended-valued rv, and the requirement \( P(S_n = 0 \text{ i.o.}) = 1 \) guarantees that \( W_k(\omega) \) is well-defined for all \( k \geq 1 \) for a.e. \( \omega \).

Now, \( T_1 = W_1 \) is clearly a stopping time. Thus, by the strong Markov property, \( T_1 \) is independent of the rv \( \tilde{S}^{(1)} \equiv \tilde{S} \) with kth coordinate \( \tilde{S}_k^{(1)} \equiv \tilde{S}_k \equiv S_{T_1+k} - S_{T_1} \) and \( \tilde{S}^{(1)} \equiv \tilde{S} \). Thus \( T_2 \) is independent of the rv \( \tilde{S}^{(2)} \) with kth coordinate \( \tilde{S}_k^{(2)} \equiv \tilde{S}_{T_2+k} - \tilde{S}_{T_2} \equiv S_{T_1+T_2+k} - S_{T_1+T_2} \). Continue with \( \tilde{S}^{(3)} \), etc. [Note the relationship to interarrival times of a Bernoulli process.] \( \square \)

Exercise 7.2 (Wald’s identity) (a) Suppose \( X_1, X_2, \ldots \) are iid with mean \( \mu \), and \( N \) is a stopping time with finite mean. Show that \( S_n \equiv X_1 + \cdots + X_n \) satisfies

\[
\mathbb{E}S_N = \mu \mathbb{E}N.
\]

(b) Suppose each \( X_k \) equals 1 or \(-1\) with probability \( p \) or \( 1 - p \) for some \( 0 < p < 1 \). Then define the rv \( N \equiv \min\{n : S_n \text{ equals } -a \text{ or } b\} \), where \( a \) and \( b \) are strictly positive integers. Show that \( N \) is a stopping time that is a.s. finite. Then evaluate the mean \( \mathbb{E}N \). [Hint. \( [N \geq k] \in \mathcal{F}(S_1, \ldots, S_{k-1}) \), and is thus independent of \( X_k \), while \( S_N = \sum_{k=1}^{\infty} X_k 1_{[N \geq k]} \).]