Statistics 581, Practice Final Exam
Wellner; 11/28/2018

This exam is to be taken without the use of any books or notes.

1. (48) points) **Define** *each* of the following terms. In each case, provide an appropriate context for your definition.
   (a) The information matrix for θ in a regular parametric model $\mathcal{P} = \{P_\theta : \theta \in \Theta \subset R^k\}$.
   (b) The efficient score function for a parameter θ_1 when $\theta = (\theta_1, \theta_2)$.
   (c) The efficient influence function $\tilde{l}_1$ for a parameter θ_1 when $\theta = (\theta_1, \theta_2)$.
   (d) The efficient influence function for $\tilde{l}_\nu$ for a differentiable parameter $q(\theta) = \nu(P_\theta)$ in a regular parametric model $\mathcal{P}$.
   (e) An asymptotically linear estimator T_n of a parameter $\nu(P)$ with influence function ψ.
   (f) A locally regular estimator T_n of a parameter $\nu(P_\theta)$.

2. (32 points) **State** *four* of the following six results, providing the appropriate (brief) context for your statement:
   (a) The (elementary) Skorokhod theorem.
   (b) The multiparameter Cramér - Rao inequality (for an unbiased estimator) $T = T(\underline{X})$ of a real-valued parameter $q(\theta) = \nu(P_\theta)$.
   (c) A result about the finite-dimensional limiting distributions of the sample quantile process $\{\sqrt{n}(\mathbb{F}_n^{-1}(t) - F^{-1}(t)) : 0 < t < 1\}$ specifying the assumption(s) carefully.
   (d) The Glivenko-Cantelli theorem.
   (e) LAN (Local Asymptotic Normality) of the local - log likelihood ratios for a regular parametric model satisfying the Cramér hypotheses).
   (f) The asymptotic behavior of the likelihood ratio statistic $2 \log \lambda_n$ for testing a simple null hypothesis $\theta = \theta_0$ versus $\theta \neq \theta_0$ under a fixed alternative P_θ with $\theta \neq \theta_0$.

3. (40 points) Suppose that $\mathcal{P} = \{P_\theta : \theta \in (0,1)\}$ where P_θ has density with respect to Lebesgue measure λ on $[0,1]$ given by

$$p_\theta(x) = 2\left\{\frac{x}{\theta}1_{[0,\theta]}(x) + \frac{1-x}{1-\theta}1_{(\theta,1]}(x)\right\}.$$

   We showed in problem 5 of problem set # 9 that

$$\dot{\mathbf{l}}_\theta(x) = -\frac{1}{\theta}1_{[0,\theta]}(x) + \frac{1}{1-\theta}1_{(\theta,1]}(x),$$
$$I(\theta) = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)},$$

and that if $X_1, \ldots, X_n$ are i.i.d. P_θ with $\theta \in (0,1)$, then a $\sqrt{n}-$consistent preliminary estimator of θ is given by $\overline{\theta}_n = 3\overline{X}_n - 1$ which satisfies

$$\sqrt{n}(\overline{\theta}_n - \theta) \rightarrow_d N(0, (1-\theta+\theta^2)/2).$$

(a) Use the above facts to suggest a one-step estimator $\check{\theta}_n$ of θ.
(b) Show that the estimator $\check{\theta}_n$ you proposed in (a) can be written as

$$\check{\theta}_n = \overline{\theta}_n - (\mathbb{F}_n(\overline{\theta}_n) - \overline{\theta}_n)$$

where $\mathbb{F}_n(x) = n^{-1}\sum_{i=1}^n 1_{[X_i \le x]}$ is the empirical distribution of the X_i's.
(c) Show by a direct argument using the result of (b) that $\check{\theta}_n \rightarrow_p \theta$. Hints: Calculate the distribution function F_θ corresponding to the density p_θ, show that $F_\theta(\theta) = \theta$, and use the Glivenko-Cantelli theorem.
(d) Show by a direct argument using the result of (b) that $\sqrt{n}(\check{\theta}_n - \theta) \rightarrow_d N(0, I(\theta)^{-1})) = N(0, \theta(1-\theta))$. Hint: Use Donsker's theorem and the delta-method applied to $g(x) = F_\theta(x) - x$.
(e) Compare the asymptotic variance of the preliminary estimator $\overline{\theta}_n$ to the asymptotic variance of the one- step estimator $\check{\theta}_n$.

4. (40 points) Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, and let $\theta = (\mu, \sigma^2) \in \Theta \equiv \mathbb{R} \times (0, \infty)$. Consider testing $H: \sigma^2 = \sigma_0^2$ where σ_0^2 is a known constant, versus $K: \sigma^2 \neq \sigma_0^2$.
(a) What are the score functions for μ and σ^2, and information matrix for θ in this model?
(b) What is the maximum likelihood estimator $\widehat{\theta}_n$ of $\theta = (\mu, \sigma^2)$ under $\theta \in \Theta$? What is the maximum likelihood estimator $\widehat{\theta}_n^0$ of $\theta = (\mu, \sigma^2)$ under $\theta \in \Theta_0$?
(c) Identify the null hypothesis H in terms of a subset Θ_0 of Θ.
(d) What are the likelihood ratio, Wald, and score (or Rao) statistics for testing H versus K? What is the limiting distribution of these three test statistics under the null hypothesis H?
(e) Suppose that $X_1, \ldots, X_n$ are i.i.d. P_θ with $\theta \notin \Theta_0$. Find the limits of $n^{-1} 2 \log \lambda_n$, $n^{-1} W_n$, $n^{-1} R_n$ (almost surely or in probability).

5. (48 points). (Poisson regression). Suppose that $(Y|Z) \sim$ Poisson$(\lambda e^{\gamma Z})$, and $Z \sim$ Bernoulli(η) on $\{0, 1\}$ You may assume that η is known and that $0 < \eta < 1$. Thus Z is a "covariate" or "predictor variable", γ is a "regression parameter" which affects the intensity of the (conditionally) Poisson variable Y, and $\theta = (\lambda, \gamma)$.
   (a) Find the information matrix for θ.
   (b) Find the information and information bound for estimation γ if the parameter λ is unknown.
   (c) Find the efficient score function and the efficient influence function for estimation of γ when λ is unknown. Interpret these geometrically in terms of the scores for γ and λ.
   (d) If we observe $X_i = (Y_i, Z_i)$, $i = 1, \ldots, n$, i.i.d. P_θ, write down the likelihood equations for the maximum likelihood estimator $\hat{\theta}_n = (\hat{\lambda}_n, \hat{\gamma}_n)$. What do our theorems tell us about the asymptotic normality of $\hat{\theta}_n$?

6. (48 points). (Poisson regression, continued).
   (a) Suggest three tests of the (composite!) null hypothesis $H : \gamma = 0$ versus $K : \gamma \neq 0$. What is the distribution of each of these three statistics under the null hypothesis and under local alternatives of the form $\gamma_n = tn^{-1/2}$?
   (b) Consider estimation of the function

$$q(\theta) = \nu(P_\theta) = E_\theta(Y|Z = 1)\,.$$

   Compute $q(\theta)$ explicitly as a function of $\theta = (\lambda, \gamma)$.
   (c) Suggest a natural empirical estimator of this conditional expectation which does not rely on the Poisson model. If this estimator is called $\tilde{\nu}_n$, show that $\tilde{\nu}_n$ is asymptotically linear and find its influence function ψ explicitly.
   (d) Find the efficient influence function $\tilde{\mathbf{l}}_\nu$ for estimation of $\nu(P_\theta)$ assuming the Poisson model.
   (e) Describe the relationship between ψ and $\tilde{\mathbf{l}}_\nu$ geometrically.