Statistics 581, Problem Set 1

Wellner; 9/26/2018

Reading: Ferguson, ACILST: Chapter 1, pages 3-7;
Van der Vaart: Chapter 2, pages 2 - 12;
581-Course Notes: skim Chapter 0; read Chapter 1.
Optional further reading: Lehmann & Casella, TPE, pages 1 - 32.

Due: Wednesday, October 3, 2018.

1. (a) The case $r = 1$ of Chebychev's Inequality is known as Markov's Inequality and is usually written $P(|X| \ge \epsilon) \le E(|X|)/\epsilon$ for an arbitrary random variable X and $\epsilon > 0$. For every $\epsilon > 1$, find a distribution for X with $E(X) = 0$ and $E|X| = 1$ that gives equality in Markov's inequality.
(b) Prove for an arbitrary random variable X and $\epsilon > 0$

$$P(|X| \ge \epsilon) \le E\left\{\frac{\cosh(X) - 1}{\cosh(\epsilon) - 1}\right\}.$$

2. Let X and Y be i.i.d. Uniform$(0, 1)$ random variables Define $U = X - Y$, $V = \max(X, Y) = X \vee Y$.

   (i) What is the range of (U, V)?

   (ii) Find the joint density function $f_{U,V}(u, v)$ of the pair (U, V). Are U and V independent?

3. Ferguson, ACILST, #6, page 7. (This is known as the Pólya-Cantelli lemma; see Chapter 2, Proposition 2.11, page 10.) Give an example of the use of this lemma. (See Lemma 2.11, p. 12, *Asymp. Statist.* for a multivariate version of this.)

4. Suppose that for $\theta \in R$,

$$f_\theta(u, v) = \{1 + \theta(1 - 2u)(1 - 2v)\}1_{[0,1]^2}(u, v).$$

(a) For what values of θ is f_θ a density function on $[0, 1]^2$?
(b) For the set of θ's you identified in (a), find the corresponding distribution function F_θ and show that it has Uniform$(0, 1)$ marginal distributions.
(c) If $(U, V) \sim F_\theta$, compute the correlation $\rho(U, V) \equiv \rho$. Does this show any difficulty with this family of distributions as a model of dependence?

5. (a) Lehmann & Casella, TPE, problem 3.5, page 64.
Let S be the support of a distribution on a Euclidean space $(\mathcal{X}, \mathcal{A})$. Then, (i) S is closed; (ii) $P(S) = 1$; (iii) S is the intersection of all closed sets C with $P(C) = 1$. (The *support* S of a distribution P on $(\mathcal{X}, \mathcal{A})$ is the set of all points x for which $P(A) > 0$ for all open rectangles $A = \{(x_1, \ldots, x_n) : \; a_i < x < b_i, \; i = 1, \ldots, n\}$ for numbers $a_i < b_i$ in R.)
(b) Lehmann & Casella, TPE, problem 3.6, page 64.
Show that if P and Q are two probability measures over the same Euclidean space which are equivalent (i.e. P is absolutely continuous with respect to Q and Q is absolutely continuous with respect to P), then they have the same support.
(c) Lehmann & Casella, TPE, problem 3.7, page 64.
Let P and Q assign probabilities

$$P: \quad P(X = 1/n) = p_n > 0, \quad n = 1, 2, \ldots \quad (\sum_n p_n = 1),$$

$$Q: \quad P(X = 0) = 1/2; \quad P(X = 1/n) = q_n > 0, \quad n = 1, 2, \ldots (\sum_n q_n = 1/2).$$

Then, show that P and Q have the same support but are not equivalent.

6. **Bonus problem 1:** Suppose that $X \sim$ Uniform$(0, 1)$ and $Y = 1 - X$. Find the joint distribution function $F(x, y) = F_{X,Y}(x, y)$ of (X, Y). Does this distribution function have a joint density with respect to Lebesgue measure?

7. **Bonus problem 2:**
(a) Let $Z_1, \ldots, Z_n$ be i.i.d. $N(0, 1)$ random variables, and let

$$\underline{U}_n = \frac{(Z_1, \ldots, Z_n)}{\sqrt{\sum_{i=1}^n Z_i^2}} = \frac{\underline{Z}}{\|\underline{Z}\|}$$

so that $\|\underline{U}_n\| = 1$. Let Γ be an orthogonal transformation of $\mathbb{R}^n$. Show that $\Gamma \underline{U}_n$ has the same distribution as $\underline{U}_n$ for every orthogonal Γ. Thus $\underline{U}_n$ is uniformly distributed over the unit sphere S^{n-1} in $\mathbb{R}^n$.
(b) Let $\underline{U}_n$ be distributed uniformly over the unit sphere S^{n-1}. Show that $\sqrt{n}(U_{n,1}, U_{n,2}) \to_d (Z_1, Z_2)$ where Z_1, Z_2 are independent $N(0, 1)$ random variables.