Statistics 581, Problem Set 1 Solutions
Wellner; 10/3/2018

1. (a) The case $r = 1$ of Chebychev's Inequality is known as Markov's Inequality and is usually written $P(|X| \ge \epsilon) \le E(|X|)/\epsilon$ for an arbitrary random variable X and $\epsilon > 0$. For every $\epsilon > 1$, find a distribution for X with $E(X) = 0$ and $E|X| = 1$ that gives equality in Markov's inequality.
(b) Prove for an arbitrary random variable X and $\epsilon > 0$

$$P(|X| \ge \epsilon) \le E\left\{\frac{\cosh(X) - 1}{\cosh(\epsilon) - 1}\right\}.$$

Solution: (a) Given $\epsilon > 1$, let $X = \pm a > 0$ with probability $1/(2\epsilon) < 1/2$ and let $X = 0$ with probability $1 - 1/\epsilon$. Then

$$\begin{aligned} E(X) &= aP(X = a) + (-a)P(X = -a) + 0 \cdot P(X = 0) \\ &= a/(2\epsilon) - a/(2\epsilon) = 0, \end{aligned}$$

$$E|X| = aP(X = a) + aP(X = -a) = a/\epsilon = 1$$

if we take $a = \epsilon$. On the other hand

$$P(|X| \ge \epsilon) = P(X \ge \epsilon) + P(-X \ge \epsilon) = 1/(2\epsilon) + 1/(2\epsilon) = 1/\epsilon,$$

so equality holds in Markov's inequality for this fixed $\epsilon > 1$.
(b) Note that $g(y) \equiv \cosh(y) - 1$ satisfies $g(0) = 0$ and $g(-y) = g(y)$. Therefore, for any $\epsilon > 0$

$$P(|X| \ge \epsilon) = P(g(X) \ge g(\epsilon)) \le \frac{Eg(X)}{g(\epsilon)}$$

where the inequality is just Markov's inequality applied to $Y = g(X)$ and $\epsilon' = g(\epsilon)$.

2. Let X and Y be i.i.d. Uniform$(0, 1)$ random variables Define $U = X - Y$, $V = \max(X, Y) = X \vee Y$.

 (i) What is the range of (U, V)?

 (ii) Find the joint density function $f_{U,V}(u, v)$ of the pair (U, V). Are U and V independent?

Solution: (i) The range of (X,Y) is
$A=\{(x,y):0\le x\le 1,\ 0\le y\le 1\}$. The range of (U,V) is

$$B \;=\; \{(u,v):0\le u\le 1, u\le v\le 1\}\cup\{(u,v):-1\le u<0,-u\le v\le 1\}.$$

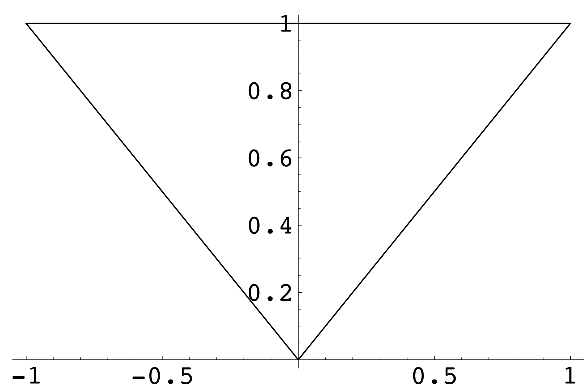

Figure 1: Range of U,V .

(ii) First solution - via Jacobians: The transformation $(X,Y) \to (U,V)$ is 1-1 and onto from A to B. On the set $x < y$, its inverse is given by $X = U+V$, $Y = V$; on the set $x > y$, its inverse is given by $X = V$, $Y = V - U$. These mappings are continuously differentiable on $B^* \equiv B \setminus \{(u,v) : (0,v)\} = B \setminus$ a null set. On B^* the Jacobian of the transformations are

$$\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \qquad \text{if } x < y, \qquad \det\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = 1 \qquad \text{if } x > y. \tag{1}$$

Thus by the usual transformation of densities formula, the joint density of (U,V) is obtained from $f_{X,Y}(x,y) = 1_{[0,1]}(x)1_{[0,1]}(y)$ as follows:

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(x(u,v),y(u,v))|\det\frac{\partial(x,y)}{\partial(u,v)}|1_{[x(u,v)<y(u,v)]} \\ &\qquad + f_{X,Y}(x(u,v),y(u,v))|\det\frac{\partial(x,y)}{\partial(u,v)}|1_{[x(u,v)>y(u,v)]} \\ &= \big(1_{[0,1]}(u+v)1_{[0,1]}(v)1_{[u+v<v]} + 1_{[0,1]}(v)1_{[0,1]}(v-u)1_{[v>v-u]}\big) \\ &= 1_B(u,v)\,. \end{aligned}$$

Thus the joint density of (U,V) is uniform on B. The random variables U and V are clearly *not* independent since the range of (U,V) is not a product set in R^2; moreover, the joint density of (U,V) does not factor into the product of its marginal densities. [The marginal densities are given by

$$f_U(u) = \int f_{U,V}(u,v)dv = \begin{cases} \int_u^1 dv = 1-u, & u \in [0,1] \\ \int_{-u}^1 dv = 1+u, & u \in [-1,0) \end{cases}$$

and

$$f_V(v) = \int f_{U,V}(u,v)du = \int_{-v}^{v} du = 2v1_{[0,1]}(v).]$$

Second solution by direction calculation of the joint distribution function: Note that we can write

$$\begin{aligned} &P(U \le u, V \le v) \\ &\quad = P(X - Y \le u, X \vee Y \le v) = P(X-Y \le u, X \le v, Y \le v) \\ &\quad = P(Y \ge X - u, X \le v, Y \le v) \\ &\quad = \begin{cases} v^2 - \frac{1}{2}(v-u)^2, & \text{if } \ 0 \le u \le v \le 1, \\ \frac{1}{2}(v+u)^2, & \text{if } \ -1 \le u < 0,\ 0 < -u \le v \le 1\,. \end{cases} \end{aligned}$$

(This is easy by pictures!) Computing $(\partial^2/\partial u \partial v) P(U \le u, V \le v)$ on each of these pieces separately again yields $f_{U,V}(u,v) = 1_B(u,v)$. Also note that the marginal distribution functions of U and V are given by $F_U(u) = (1/2)(1+u)^2 1_{[-1,0)}(u) + \{1 - \frac{1}{2}(1-u)^2\} 1_{[0,1]}(u)$ on $-1 \le u \le 1$ and $F_V(v) = v^2$ for $0 \le v \le 1$.

3. Ferguson, ACILST, #6, page 7. (a) (This is known as the Pólya-Cantelli lemma; see Chapter 2, Proposition 2.11, page 10.)
(b) Give an example of the use of this lemma.
(See Lemma 2.11, p. 12, *Asymp. Statist.* for a multivariate version of this.)

Solution. (a) For the proof, see Ferguson, ACILST page 173. See van der Vaart (1998), page 12, for a sketch of the proof in the multivariate case.
(b) As an example, suppose that a test statistic T_n is assumed to have a t_{n-1} distribution under a null hypothesis, $\alpha \in (0, 1/2)$ and we reject the hull hypothesis H_0 if $T_n \ge t_{n-1,\alpha}$. If in fact T_n is not exactly t_{n-1} distributed, but we do have $T_n \to_d Z \sim N(0,1)$ under the null hypothesis, what is the asymptotic size of the test? That is, find the limit of $P(T_n \ge t_{n-1,\alpha})$ under these assumptions. Claim: this is exactly α. Let $\Phi(z) = \int_{-\infty}^{z} (2\pi)^{-1/2} \exp(-y^2/2) dy$. Then

$$\begin{aligned} P(T_n \ge t_{n-1}) &= (1 - H_n(t_{n-1})) \\ &= (1 - H_n(t_{n-1,\alpha})) - P(Z \ge t_{n-1,\alpha}) + P(Z \ge t_{n-1,\alpha}) \\ &= -(H_n(t_{n-1,\alpha}) - \Phi(t_{n-1,\alpha})) + (1 - \Phi(t_{n-1,\alpha})) \\ &\to 0 + 1 - \Phi(z_\alpha) = \alpha \end{aligned}$$

where the convergence in the first term follows from the Pólya- Cantelli lemma and the convergence in the second term follows from $t_{n-1,\alpha} \to z_\alpha$ where z_α satisfies $\Phi(z_\alpha) = 1 - \alpha$.

4. Suppose that for $\theta \in R$,

$$f_\theta(u,v) = \{1 + \theta(1-2u)(1-2v)\} 1_{[0,1]^2}(u,v).$$

(a) For what values of θ is f_θ a density function on $[0,1]^2$?
(b) For the set of θ's you identified in (a), find the corresponding distribution function F_θ and show that it has Uniform$(0,1)$ marginal distributions.

(c) If $(U,V) \sim F_\theta$, compute the correlation $\rho(U,V) \equiv \rho$. Does this show any difficulty with this family of distributions as a model of dependence?

Solution: (a) For f_θ to be a density function, we must have $f_\theta(u,v) \geq 0$ for all $(u,v) \in [0,1]^2$ and

$$\int_0^1 \int_0^1 f_\theta(u,v)dudv = 1\,. \tag{2}$$

Now

$$\int_0^1 \int_0^1 f_\theta(u,v)dudv = 1 + \theta \int_0^1 \int_0^1 (1-2u)(1-2v)dudv = 1$$

for all $\theta \in R$ since

$$\int_0^1 \int_0^1 (1-2u)(1-2v)dudv = \int_0^1 (1-2u)du \int_0^1 (1-2v)dv = 0 \cdot 0 = 0\,,$$

and hence (2) holds for all θ. The requirement that f_θ be non-negative is just

$$1 + \theta(1-2u)(1-2v) \geq 0 \qquad \text{for all } \ (u,v) \in [0,1]^2\,,$$

or equivalently that

$$\theta(1-2u)(1-2v) \geq -1 \qquad \text{for all } \ (u,v) \in [0,1]^2\,.$$

By monotonicity of $1-2u$, this holds if and only if it holds for $(u,v) \in \{(0,0),(0,1),(1,0),(1,1)\}$; i.e.

$$\theta \geq -1, \qquad -\theta \geq -1, \qquad -\theta \geq -1, \qquad \text{and } \ \theta \geq -1\,.$$

Thus it follows that f_θ is a density function for $\theta \in [-1,1]$, or $|\theta| \leq 1$.
(b) The corresponding distribution function F_θ is given by

$$\begin{aligned}
F_\theta(u,v) &= \int_0^u \int_0^v f_\theta(r,s)drds \\
&= \int_0^u \int_0^v \{1 + \theta(1-2r)(1-2s)\}drds \\
&= uv + \theta \int_0^u (1-2r)dr \int_0^v (1-2s)ds \\
&= uv + \theta u(1-u)v(1-v) \\
&= uv\,\{1 + \theta(1-u)(1-v)\}\,.
\end{aligned}$$

Note that

$$F_\theta(u,1) = u\,, \quad \text{and} \quad F_\theta(1,v) = v\,,$$

so F_θ has Uniform(0, 1) marginal distributions.
(c) It follows from part (iv) of Proposition 1.4.1, page 20, Chapter 1, that (by taking $G(x) = x$, $H(x) = x$))

$$\begin{aligned} Cov(U,V) &= \int_0^1 \int_0^1 \{F_\theta(u,v) - uv\} du dv \\ &= \int_0^1 \int_0^1 \theta u(1-u)v(1-v) du dv \\ &= \theta \left(\int_0^1 u(1-u) du \right)^2 \\ &= \frac{1}{36}\theta \end{aligned}$$

since

$$\int_0^1 u(1-u)du = \frac{1}{2}u^2 - \frac{1}{3}u^3 \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\,.$$

Now since $Var(U) = Var(V) = 1/12$ (since they are both Uniform(0, 1)), it follows that

$$\rho(U,V) = \frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} = \frac{\theta/36}{\sqrt{(1/12)(1/12)}} = \frac{\theta}{3}\,.$$

Note that this implies that $|\rho(U,V)| \le 1/3$, and hence this family of distributions does not include any distributions on $[0,1]^2$ with correlations larger than 1/3 in absolute value.

5. (a) Lehmann & Casella, TPE, problem 3.5, page 64.
Let S be the support of a distribution on a Euclidean space $(\mathcal{X}, \mathcal{A})$. Then, (i) S is closed; (ii) $P(S) = 1$; (iii) S is the intersection of all closed sets C with $P(C) = 1$. (The *support* S of a distribution P on $(\mathcal{X}, \mathcal{A})$ is the set of all points x for which $P(A) > 0$ for all open rectangles $A = \{(x_1, \ldots, x_n) :\ a_i < x < b_i,\ i = 1, \ldots, n\}$ for numbers $a_i < b_i$ in R.)
(b) Lehmann & Casella, TPE, problem 3.6, page 64.
Show that if P and Q are two probability measures over the same Euclidean space which are equivalent (i.e. P is absolutely continuous

with respect to Q and Q is absolutely continuous with respect to P), then they have the same support.
(c) Lehmann & Casella, TPE, problem 3.7, page 64.
Let P and Q assign probabilities

$$P: \quad P(X = 1/n) = p_n > 0, \quad n = 1, 2, \ldots \quad (\sum_n p_n = 1),$$

$$Q: \quad P(X = 0) = 1/2; \quad P(X = 1/n) = q_n > 0, \quad n = 1, 2, \ldots (\sum_n q_n = 1/2).$$

Then, show that P and Q have the same support but are not equivalent.

Solution: (a) (i) Suppose that S is not closed. Then there exists a sequence $\{x_n\} \subset S$ such that $x_n \to x_0 \in S^c$. But then, for every $\epsilon > 0$ there is an open ball $B(x_0, \epsilon)$ such that $x_n \in B(x_0, \epsilon)$ for $n \geq N_\epsilon$. Since each x_n is a support point, $P(B(x_0, \epsilon)) > 0$ for each $\epsilon > 0$. But for any open set A with $x_0 \in A$, $B(x_0, \epsilon) \subset A$ for some $\epsilon > 0$, and hence $P(A) \geq P(B(x_0, \epsilon) > 0$. But this implies $x_0 \in S$. Contradiction. Thus S is closed.
(ii) $P(S) = 1$. From (i) S is closed, so S^c is open. Since $x \in S^c$ if and only if $x \in A_x$ with A_x an open rectangle satisfying $P(A_x) = 0$. Thus $S^c \subset \cup_x A_x$. By the Lindelöf theorem, for any such open covering $\{A_x\}_{x \in S^c}$ of $S^c \subset R^d$, there is a countable subcollection $\{A_{x_n}\}$ which covers S^c: $S^c \subset \cup_n A_{x_n}$. Then we have

$$P(S^c) \leq P(\cup_n A_{x_n}) \leq \sum_n P(A_{x_n}) = \sum_n 0 = 0\,.$$

Hence $P(S) = 1$.
(iii) We want to show that $S = \cap\{C : \ C \text{ closed}, \ P(C) = 1\}$. From (i) and (ii) we know that S is in the collection of sets on the right side, so it follows that $S \supset \cap\{C : \ C \text{ closed}, P(C) = 1\}$. Thus it remains to show that $S \subset \cap\{C : \ C \text{ closed}, P(C) = 1\}$. Equivalently, it remains to show that $S^c \supset \cup\{C^c : C^c \text{ open}, \ P(C^c) = 0\}$. But if $x \in \cup\{C^c : C^c \text{ open}, \ P(C^c) = 0\}$, then $x \in C^c$ for some C^c open with $P(C^c) = 0$, and hence also $x \in A \subset C^c$ for some open rectangle A (an open ball centered at x for the metric $\|y\| = \max_{1 \leq i \leq d} |x_i|$) with $P(A) \leq P(C^c) = 0$. Hence $x \in S^c$.

(b) Suppose that P and Q are equivalent: i.e. $Q \prec\prec P$ and $P \prec\prec Q$. Then for any open set A, $P(A) = 0$ if and only if $Q(A) = 0$. This implies that for any closed set A^c,

$$P(A^c) = 1 \qquad \text{if and only if} \qquad Q(A^c) = 1\,.$$

This implies that the minimal closed set S_P with $P(S_P) = 1$ is also the minimal closed set S_Q with $Q(S_Q) = 1$; i.e. $S_P = \text{supp}(P) = \text{supp}(Q) = S_Q$.
(c) Since $P(X = 1/n) = p_n > 0$ for $n = 1, 2, \ldots$ with $\sum_1^\infty p_n = 1$, it follows that $\text{supp}(P) = \{0, \ldots, 1/n, \ldots, 1/2, 1\}$, which is closed. Similarly, Since $Q(X = 1/n) = q_n > 0$ for $n = 1, 2, \ldots$ with $\sum_1^\infty q_n = 1/2$, and $Q(X = 0) = 1/2$, it follows that $\text{supp}(Q) = \{0, \ldots, 1/n, \ldots, 1/2, 1\} = \text{supp}(P)$. But $P(\{0\}) = 0$ while $Q(\{0\}) = 1/2$, so $Q \prec\prec P$ fails. Thus Q and P are not equivalent.