Statistics 581, Problem Set 3 Solutions

Wellner; 10/18/2018

1. Ferguson, ACILST, page 34, problem 1(a) (modified slightly):
Suppose that $X_1, X_2, \ldots$ are i.i.d. in R^2 with distribution giving probability θ_1 to $(1,0)'$, probability θ_2 to $(0,1)'$, θ_3 to $(-1,0)'$ and θ_4 to $(0,-1)'$ where $\theta_j \ge 0$ for $j = 1,2,3,4$ and $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 1$.
(a) Find $\mu = E(X_1)$.
(b) Compute $E(X_1 X_1^T)$ and $\Sigma = E(X_1 - \mu)(X_1 - \mu)^T$.
(c) Find the limiting distribution of $\sqrt{n}(\overline{X}_n - \mu)$ and describe the resulting approximation to the distribution of $\overline{X}_n$.
(d) Find values of $(\theta_1, \ldots, \theta_4)$ such that Σ has rank 1 and $\det(\Sigma) = 0$.

Solution: (a) The mean vector μ is

$$\mu = E(X_1) \;=\; \theta_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \theta_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \theta_3 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \theta_4 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \theta_1 - \theta_3 \\ \theta_2 - \theta_4 \end{pmatrix}.$$

(b) The second moment matrix $E(XX^T)$ is

$$\begin{aligned} E(XX^T) &= \theta_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \theta_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \theta_3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \theta_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \theta_1 + \theta_3 & 0 \\ 0 & \theta_2 + \theta_4 \end{pmatrix}, \end{aligned}$$

and hence the covariance matrix $\Sigma = E(XX^T) - E(X)E(X)^T$ is

$$\begin{aligned} \Sigma &= \begin{pmatrix} \theta_1 + \theta_3 & 0 \\ 0 & \theta_2 + \theta_4 \end{pmatrix} - \begin{pmatrix} (\theta_1 - \theta_3)^2 & (\theta_1 - \theta_3)(\theta_2 - \theta_4 \\ (\theta_1 - \theta_3)(\theta_2 - \theta_4) & (\theta_2 - \theta_4)^2 \end{pmatrix} \\ &= \begin{pmatrix} \theta_1 + \theta_3 - (\theta_1 - \theta_3)^2 & -(\theta_1 - \theta_3)(\theta_2 - \theta_4) \\ -(\theta_1 - \theta_3)(\theta_2 - \theta_4) & (\theta_2 + \theta_4) - (\theta_2 - \theta_4)^2 \end{pmatrix} \end{aligned}$$

(c) By the multivariate CLT, the limiting distribution of $\sqrt{n}(\overline{X}_n - \mu)$ is $N_2(0\Sigma)$. Thus the approximating normal distribution of $\overline{X}_n$ is centered at μ with covariance matrix $n^{-1}\Sigma$.
(d) The determinant of Σ is given by

$$\det(\Sigma) = \left\{(\theta_1 + \theta_3) - (\theta_1 - \theta_3)^2\right\} \left\{(\theta_2 + \theta_4) - (\theta_2 - \theta_4)^2\right\} - (\theta_1 - \theta_3)^2 (\theta_2 - \theta_4)^2.$$

This equals 0 if $\theta_2 = \theta_4 = 0$ or if $\theta_1 = \theta_3 = 0$. In the first case

$$\Sigma = \begin{pmatrix} \theta_1 + \theta_3 - (\theta_1 - \theta_3)^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 - (\theta_1 - \theta_3)^2 & 0 \\ 0 & 0 \end{pmatrix},$$

and in the second case

$$\Sigma = \begin{pmatrix} 0 & 0 \\ 0 & (\theta_2 + \theta_4) - (\theta_2 - \theta_4)^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 - (\theta_2 - \theta_4)^2 \end{pmatrix}.$$

2. (Van der Vaart, page 24)
(a) Suppose that X_n and Y_n are independent random vectors with $X_n \to_d X$ and $Y_n \to_d Y$. Show that $(X_n, Y_n) \to_d (X, Y)$ where X and Y are independent.
(b) Suppose that $P(X_n = i/n) = 1/n$ for $i = 1, 2, \ldots, n$. show that $X_n \to_d X \sim$ Uniform$(0, 1)$.
(c) Consider the X_n's as in (b). Show that there exists a Borel set B such that $P(X_n \in B) = 1$ but $P(X \in B) = 0$. In particular, with $P_n = \mathcal{L}(X_n)$ and $P = \mathcal{L}(X)$, $d_{TV}(P_n, P) = 1$ for each n.

Solution:(a) Since $X_n \to_d X$ in $\mathbb{R}^m$ and $Y_n \to_d Y$ in $\mathbb{R}^k$ (as random vectors), we know that their characteristic functions also converge pointwise: for every $s \in \mathbb{R}^m$, $t \in \mathbb{R}^k$,

$$\begin{aligned} \phi_{X_n}(s) = Ee^{is^T X_n} &\to Ee^{is^T X} = \phi_X(s) \quad \text{and} \\ \phi_{Y_n}(t) = Ee^{it^T Y_n} &\to Ee^{it^T Y} = \phi_Y(t). \end{aligned}$$

Then, since X_n and Y_n are independent,

$$\begin{aligned} \phi_{X_n, Y_n}(s, t) &= Ee^{is^T X_n + t^T Y_n} = Ee^{is^T X_n} Ee^{it^T Y_n} \\ &\to \phi_X(s) \cdot \phi_Y(t). \end{aligned}$$

This implies that $(X_n, Y_n) \to_d (X, Y)$ where X and Y are independent.
(b) Note that for each $x \in [0, 1]$

$$F_n(x) = P(X_n \leq x) = \lfloor nx \rfloor / n \to x = P(X \leq x)$$

It follows that $X_n \to X \sim$ Uniform$(0, 1)$.
(c) Let $B = \{1/n, \ldots, n/n = 1\}$. Then $P(X_n \in B) = 1$ but $P(X \in B) = 0$. Thus $d_{TV}(P_n, P) = 1$.

3. ACILST, page 34, problem 5: Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with mean μ and variance $\sigma^2 < \infty$. Let $T_n = \sum_{j=1}^n z_{nj} X_j$ where the $\{z_{nj}\}_{j=1}^n$ are given numbers. Let $\mu_n = E(T_n)$ and $\sigma_n^2 = Var(T_n)$. Use the Lindeberg-Feller central limit theorem to show that $(T_n - \mu_n)/\sigma_n \to_d Z \sim N(0, 1)$ if $\max_{1 \leq j \leq n} z_{nj}^2 / \sum_{j=1}^n z_{nj}^2 \to 0$ as $n \to \infty$.

Solution: Let $Y_i \equiv X_i - \mu$ so that $E(Y_i) = 0$ and $Var(Y_i) = \sigma^2$. We also let $X_n i \equiv z_{ni} Y_i$ so that $T_n - \mu_n = \sum_{i=1}^n X_{ni}$, and note that $\sigma_n^2 = Var(T_n) = \sigma^2 \sum_{i=1}^n z_{ni}^2$. To prove the claimed asymptotic normality we need to check that the Lindeberg-condition holds. For $\epsilon > 0$ we want to show that

$$LF_n(\epsilon) = \frac{1}{\sigma_n^2} \sum_{i=1}^n E\left\{X_{ni}^2 1\{|X_{ni}| > \epsilon\sigma_n\right\} \to 0$$

for every $\epsilon > 0$. But we note that

$$LF_n(\epsilon) = \frac{1}{\sigma_n^2} \sum_{i=1}^n E\left\{ z_{ni}^2 Y_i^2 1\{|z_{ni}||Y_i| > \epsilon\sigma \left(\sum_{i=1}^n z_{ni}^2\right)^{1/2} \} \right\}$$

$$
\begin{aligned}
&\le \frac{1}{\sigma_n^2}\sum_{i=1}^n z_{ni}^2 E\left\{ Y_i^2 1\{\max_{1\le i\le n}|z_{ni}||Y_i| > \epsilon\sigma\left(\sum_{i=1}^n z_{ni}^2\right)^{1/2}\}\right\} \\
&= \frac{1}{\sigma^2} E\left\{ Y_1^2 1\{|Y_1|^2 > \epsilon\sigma\sqrt{\sum_{i=1}^n z_{ni}^2 / \max_{1\le i\le n}|z_{ni}|}\}\right\} \\
&\to 0
\end{aligned}
$$

by the dominated convergence theorem with dominating function Y_1^2 since $E(Y_1^2) < \infty$.

4. (a) ACILST, problem 4, page 49: Let $X_1, \ldots, X_n$ be a sample of size n from the beta distribution Beta $(\theta, 1)$ with $\theta > 0$. Show that the method of moments estimate of θ is $\hat\theta_n = \overline{X}_n/(1-\overline{X}_n)$.
(b) Find the asymptotic distribution of $\hat\theta_n$.
(c) Is $\hat\theta_n$ asymptotically linear? If so, find the influence function of $\hat\theta_n$.
(d) Find the Cramér-Rao lower bound for estimation of θ and compare it to the asymptotic variance you found in (b).

Solution: (a) If $Y \sim \text{Beta}(\alpha, \beta)$, then $E(Y) = \alpha/(\alpha+\beta)$, so $E_\theta X_1 = \theta/(\theta+1)$. Thus the method of moments estimator $\hat\theta_n$ of θ satisfies $\overline{X}_n = \hat\theta_n/(\hat\theta_n + 1)$. Thus $\hat\theta_n = \overline{X}_n/(1-\overline{X}_n)$ as claimed.
(b) Now $Var(Y) = \alpha\beta/((\alpha+\beta)^2(\alpha+\beta+1))$, so $Var_\theta(X_1) = \theta/((\theta+1)^2(\theta+2))$. Thus $\sqrt{n}(\overline{X}_n - E_\theta(X_1)) \to_d N(0, Var_\theta(X_1))$, and by the delta method with $g(y) \equiv y/(1-y)$ we find that

$$
\sqrt{n}(\hat\theta_n - \theta) \to_d \dot g(E_\theta(X_1)) N\left(0, \frac{\theta}{(\theta+1)^2(\theta+2)}\right) = N\left(0, \frac{\theta(\theta+1)^2}{(\theta+2)}\right) \equiv N(0, V_\theta^2).
$$

(c) Note that the density f_θ of one observation is $f_\theta(x) = \theta x^{\theta-1} 1_{(0,1)}(x)$. Thus $\log f_\theta(x) = \log\theta + (\theta-1)\log x$, and the score function for θ is $\dot{\mathbf{l}}_\theta(x) = \theta^{-1} + \log x$. This yields $I_\theta = E\left\{-\ddot{\mathbf{l}}_{\theta\theta}(X_1)\right\} = \theta^{-2}$. Thus the Cramér - Rao bound for unbiased estimators $\tilde\theta_n$ of θ is given by $Var_\theta(\sqrt{n}(\tilde\theta_n - \theta)) \ge \theta^2 \equiv CRLB(\theta)$. Note that this is strictly smaller than the asymptotic variance of the method of moments estimator $\hat\theta_n$ found in (b). In fact $\hat\theta_n$ is severely inefficient for small values of θ since the ratio $CRLB(\theta)/V_\theta^2 \to 0$ as $\theta \to 0$; see Figure 1.
(d) From the extended delta method discussed in class on 10 October, with $g(v) = v/(1-v)$, $g'(v) = 1/(1-v)^2$ it follows that

$$
\sqrt{n}\left(g(\overline{X}_n) - g(E_\theta(X)\right) - g'(E_\theta(X))(\overline{X}_n - E_\theta(X)) = o_p(1).
$$

Since $g(\overline{X}_n) = \hat\theta_n$ and $g(E_\theta(X)) = \theta$ it follows that

$$
\begin{aligned}
\sqrt{n}(\hat\theta_n - \theta) &= \frac{1}{(1-E_\theta(X))^2}\frac{1}{\sqrt{n}}\sum_{i=1}^n (X_i - E_\theta(X)) + o_p(1) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^n (1+\theta)^2 (X_i - E_\theta(X)) + o_p(1).
\end{aligned}
$$

Thus $\hat{\theta}_n$ is asymptotically linear with influence function

$$\psi(x) \equiv \psi_\theta(x) = (1+\theta)^2(x - E_\theta(X)).$$

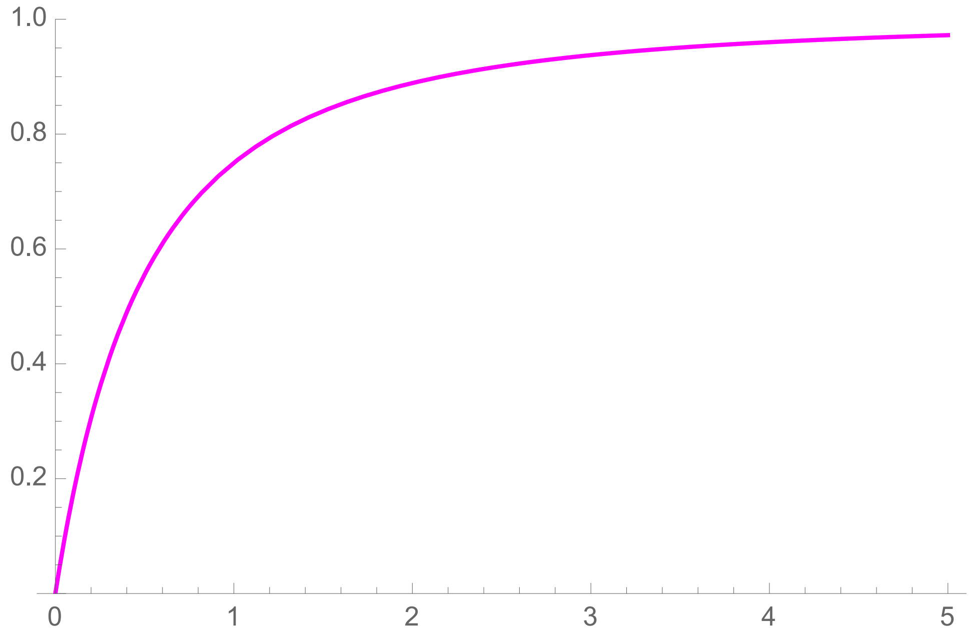

Figure 1: Ratio $CRLB(\theta)/V_\theta^2$ as a function of θ

5. Suppose that $X_1, \ldots, X_n$ are i.i.d. with $E(X_1) = \mu$, $Var(X_1) = \sigma^2 < \infty$, and $E|X_1|^6 < \infty$. Let $M_{j,n} \equiv n^{-1}\sum_{i=1}^n (X_i - \overline{X}_n)^j$ be the $j-$th sample central moment for $j \in \{2, 3, \ldots\}$.
(a) ACILST, page 49, problem 3.
(b) Find the joint asymptotic distribution of $\sqrt{n}((\overline{X}_n, M_{2,n}, M_{3,n})^T - (\mu, \sigma^2, m_3)^T)$ where $m_3 \equiv E(X_1 - \mu)^3$ is the population 3rd central moment.
(c) Find the asymptotic distribution of $\sqrt{n}(\kappa_{3,n} - \kappa_3)$ where $\kappa_{3,n} \equiv M_{3,n}/M_{2,n}^{3/2}$ is the sample skewness and $\kappa_3 \equiv m_3/\sigma^3$ is the population skewness. (See vdV Example 3.5, page 29.)

Solution: (a) $M_{3,n}$ is location invariant, so we may assume without loss of generality that $\mu = E(X_1) = 0$. As a first step, note that

$$\begin{aligned} M_{3,n} &= n^{-1}\sum_{i=1}^n (X_i^3 - 3X_i^2 \cdot \overline{X}_n + 3X_i\overline{X}_n - \overline{X}_n^3) \\ &= \overline{X^3}_n - 3\overline{X^2}_n + 2\overline{X}_n^3 \\ &\to_p E(X^3) - 3E(X^2) \cdot E(X) + 2(E(X))^3 \\ &= E(X^3) - 0 + 0 = E(X^3) \end{aligned}$$

by the WLLN and the Mann-Wald theorem. Now by the multivariate CLT, assuming that $E(X^6) < \infty$

$$\sqrt{n}\begin{pmatrix} \overline{X}_n - E(X) \\ \overline{X^2}_n - E(X^2) \\ \overline{X^3}_n - E(X^3) \end{pmatrix} \to_d \underline{Z} \sim N_3(0, \Sigma)$$

where, with $\mu_k \equiv E(X^k)$,

$$\Sigma = \begin{pmatrix} \mu_2 & \mu_3 & \mu_4 \\ \mu_3 & \mu_4 - \mu_2^2 & \mu_5 - \mu_2\mu_3 \\ \mu_4 & \mu_5 - \mu_2\mu_3 & \mu_6 - \mu_3^2 \end{pmatrix}.$$

Furthermore,

$$\sqrt{n}(M_{3,n} - m_3) \quad = \quad \sqrt{n}(g(\overline{X}_n, \overline{X^2}_n, \overline{X^3}_n) - g(0, \mu_2, \mu_3))$$

where

$$g(u, v, w) = w - 3vu + 2u^3$$

is differentiable at $(u, v, w) = (0, \mu_2, \mu_3)$ with

$$g'(u, v, w) = (-3v + 6u^2, -3u, 1), \quad \text{so that} \quad g'(0, \mu_2, \mu_3) = (-3\mu_2, 0, 1).$$

It follows from the delta-method (or $g-$prime theorem) that, with $\sigma^2 \equiv \mu_2$,

$$\begin{aligned} \sqrt{n}(M_{3,n} - m_3) &= \sqrt{n}(g(\overline{X}_n, \overline{X^2}_n, \overline{X^3}_n) - g(0, \mu_2, \mu_3)) \\ &\to_d (-3\mu_2, 0, 1)\underline{Z} = Z_3 - 3\mu_2 Z_1 \\ &\sim N(0, \mu_6 + 9\sigma^6 - 6\sigma^2\mu_4). \end{aligned}$$

Note that when $X \sim N(0, \sigma^2)$ with $\mu_4 = 3\sigma^4$ this becomes $N(0, 6\sigma^6)$.
(b) Now we regard g in (a) above as g_3 where $\underline{g} = (g_1, g_2, g_3)^T$ with

$$g_1(u, v, w) = u, \qquad g_2(u, v, w) = v - u^2, \qquad g_3(u, v, w) = w - 3vu + 2u^3.$$

Thus

$$\begin{aligned} \nabla g_1(u, v, w) &= (1, 0, 0)^T, \\ \nabla g_2(u, v, w) &= (-2u, 1, 0)^T, \\ \nabla g_3(u, v, w) &= (-3v + 6u^3, -3u, 1)^T, \end{aligned}$$

and we find that

$$g'(0, \mu_2, \mu_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3\mu_2 & 0 & 1 \end{pmatrix}.$$

Thus by the delta-method

$$\sqrt{n}\begin{pmatrix} \overline{X}_n - 0 \\ M_{2,n} - \sigma^2 \\ M_{3,n} - m_3 \end{pmatrix} \to_d \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 - 3\sigma^2 Z_1 \end{pmatrix} \equiv \underline{\widetilde{Z}} \sim N_3(0, \widetilde{\Sigma})$$

where

$$\begin{aligned} \widetilde{\Sigma} &= \begin{pmatrix} \sigma^2 & \mu_3 & \mu_4 - 3\sigma^4 \\ \mu_3 & \mu_4 - \sigma^4 & \mu_5 - 3\sigma^2\mu_3 \\ \mu_4 - 3\sigma^4 & \mu_5 - 3\sigma^2\mu_3 & \mu_6 + 9\sigma^6 - 6\sigma^2\mu_4 \end{pmatrix} \\ &= \begin{pmatrix} \sigma^2 & \mu_3 & \sigma^4\gamma_2 \\ \mu_3 & \sigma^4(2 + \gamma_2) & \sigma^5(\mu_5/\sigma^5 - 3\kappa_3) \\ \sigma^4\gamma_2 & \sigma^5(\mu_5/\sigma^5 - 3\kappa_3) & \sigma^6(\mu_6/\sigma^6 - 9 - 6\gamma_2 \end{pmatrix}. \end{aligned}$$

Note that $\mu_4 - 3\sigma^4 = \sigma^4(\mu_4/\sigma^4 - 3) = \sigma^4\gamma_2$ in the notation of the 581 course notes, example 2.3.2 (so that $\mu_3 - \sigma^4 = \sigma^4(\mu_3/\sigma^4 - 1) = \sigma^4(2 + \gamma_2)$), while

$$\begin{aligned} \mu_5 - 3\sigma^2\mu_3 &= \sigma^5(\mu_5/\sigma^5 - 3\mu_3/\sigma^3), \\ \mu_6 + 9\sigma^6 - 6\sigma^2\mu_4 &= \sigma^6(\mu_6/\sigma^6 + 9 - 6\mu_4/\sigma^4). \end{aligned}$$

(c) To find the limiting distribution of $\kappa_{3,n}$ we use the delta method again in combination with the result of (b): now $g(u, v, w) = w/v^{3/2}$ is differentiable at $v \neq 0$ with derivative $g'(u, v, w) = (0, (-3/2)v^{-5/2}w, v^{-3/2})$ so that

$$g'(0, \sigma^2, m_3) = (0, -(3/2)\sigma^{-2}(m_3/\sigma^3), \sigma^{-3}) = (0, -(3/2)\sigma^{-2}\kappa_3, \sigma^{-3}).$$

Thus it follows from the delta method that

$$\begin{aligned} \sqrt{n}(\kappa_{3,n} - \kappa_3) &= \sqrt{n}(g(\overline{X}_n, M_{2,n}, M_{3,n}) - g(0, \sigma^2, m_3)) \\ &\to_d \sigma^{-3}\widetilde{Z}_3 - (3/2)\sigma^{-2}\kappa_3\widetilde{Z}_2 \\ &\sim N(0, V^2) \end{aligned}$$

where

$$V^2 = \frac{\mu_6}{\sigma^6} - 9 - 6\gamma_2 + \frac{9}{4}\kappa_3^2(2 + \gamma_2) - 3\kappa_3\left(\frac{\mu_5}{\sigma^5} - 3\kappa_3\right).$$

Note that when $X \sim N(0, \sigma^2)$ we have $\kappa_3 = 0$, $\gamma_2 = 0$ and $\mu_6/\sigma^6 = 15$, so that $V^2 = 15 - 9 = 6$. Thus if $X \sim N(\mu, \sigma^2)$ the test "reject $H : X \sim N(\mu, \sigma^2)$ in favor of $K_{skew}: \ \ \kappa_3 \neq 0$ if $|\sqrt{n}\kappa_{3,n}| > \sqrt{6}z_{\alpha/2}$" has approximate size α for large n; i.e. $P_{norm}(|\sqrt{n}\kappa_{3,n}| > \sqrt{6}z_{\alpha/2}) \to P(|\sqrt{6}Z| > \sqrt{6}z_{\alpha/2}) = \alpha$ as $n \to \infty$.

The following development was not part of the problem as stated, but I discussed it in class on 17 October. If the distribution function F of X has $\kappa_3 \neq 0$, then (assuming that $E|X|^6 < \infty$)

$$P_F(|\sqrt{n}\kappa_{3,n}| > \sqrt{6}z_{\alpha/2}) = P_F(|\sqrt{n}(\kappa_{3,n} - \kappa_3) + \sqrt{n}\kappa_3| > \sqrt{6}z_{\alpha/2}) \to 1$$

as $n \to \infty$ since $|\sqrt{n}\kappa_3| \to \infty$ and $|\sqrt{n}(\kappa_{3,n} - \kappa_3)| = O_p(1)$. What about the (local asymptotic power) of this test? For example, what if $X \sim F_\alpha$ with density $f(x;\alpha) = 2\phi(x)\Phi(\alpha x)$? This f is the skew-normal family of densities. Figure 2 shows the densities $f(x;\alpha)$ for $\alpha \in \{0, 1, 3, 10\}$ (in blue, magenta, purple, and green).

It turns out that the skewness of this family is given by

$$\kappa_3(F_\alpha) = \sqrt{2}(4 - \pi)\alpha^3/(\pi + (\pi - 2)\alpha^2)^{3/2};$$

see Azzalini (2014), *The Skew-Normal and Related Families*, pages 30 - 31. Figure 2 shows the densities $f(x;\alpha)$ for $\alpha \in \{0, 1, 3, 10\}$ (in blue, magenta, purple, and green). This function of α has first two derivatives equal to zero at $\alpha = 0$, and it seems natural to reparametrize by $\alpha(\beta) = \beta^{1/3}$. See Figures 3 and 4. Then with

$$\kappa_3(\tilde{F}_\beta) = \kappa_3(F_{\beta^{1/3}}) = \frac{\sqrt{2}(4 - \pi)\beta}{(\pi + (\pi - 2)\beta^{2/3})^{3/2}}$$

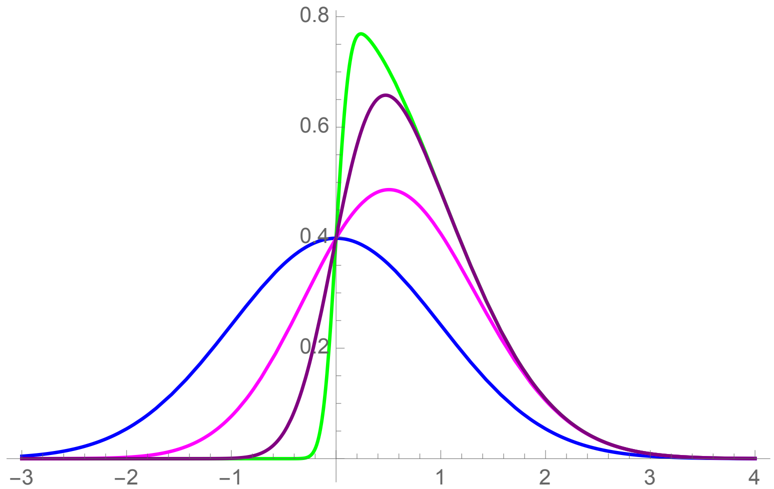

Figure 2: The densities $f(x;\alpha)$ for $\alpha \in \{0, 1, 3, 10\}$

it follows that

$$\sqrt{n}\kappa_3(\tilde{F}_{b/\sqrt{n}}) = \frac{\sqrt{2}(4-\pi)b}{(\pi + (\pi-2)(b/n^{1/2})^{2/3})^{3/2}} \to \frac{\sqrt{2}(4-\pi)b}{\pi^{3/2}}.$$

Then, modulo an argument using a triangular array multivariate CLT or reasoning via contiguity theory (see Chapter 3),

$$\begin{aligned} & P_{\tilde{F}_{b/\sqrt{n}}}(|\sqrt{n}\kappa_{3,n}| > \sqrt{6}z_{\alpha/2}) \\ &= P_{\tilde{F}_{b/\sqrt{n}}}(|\sqrt{n}(\kappa_{3,n} - \kappa_3(\tilde{F}_{b/n^{1/2}})) + \sqrt{n}\kappa_3(\tilde{F}_{b/n^{1/2}})| > \sqrt{6}z_{\alpha/2}) \\ &\to P(|\sqrt{6}Z + \sqrt{2}(4-\pi)b/(\pi-2)^{3/2}| > \sqrt{6}z_{\alpha/2}). \end{aligned}$$

Thus we can approximate the power of the skewness test of normality for these particular skew-normal (local) alternatives.

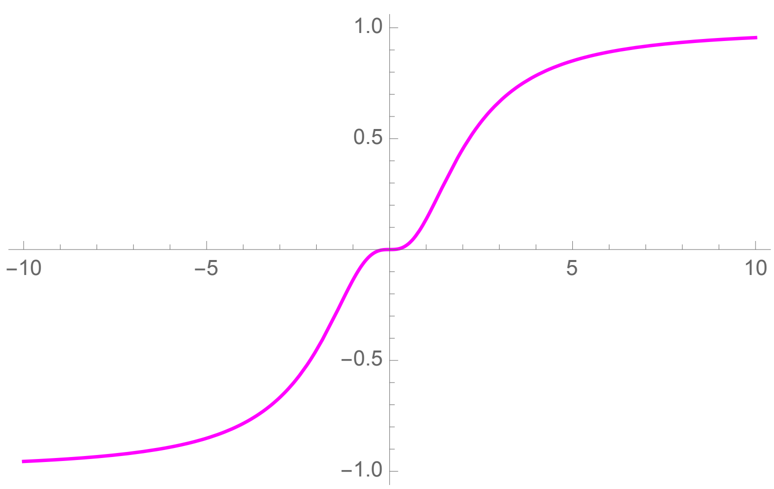

Figure 3: The function $\alpha \mapsto \kappa_3(F_\alpha)$

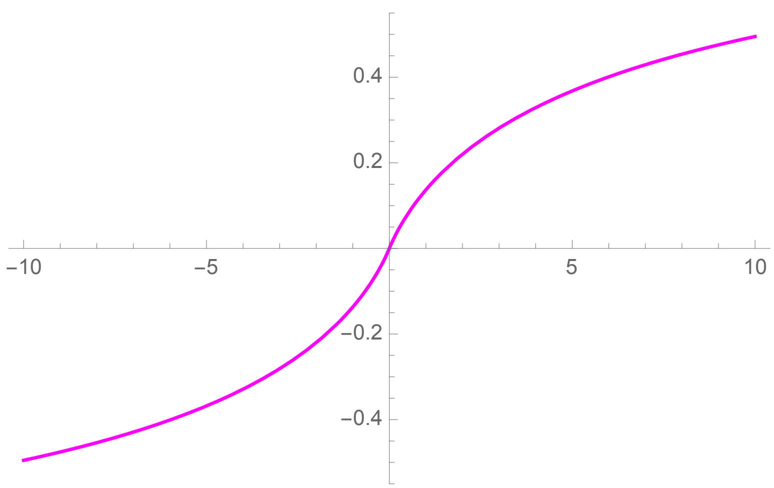

Figure 4: The function $\beta \mapsto \kappa_3(\tilde{F}_\beta)$