

Statistics 582, Final Exam

Wellner; 3/12/2007

Instructions: This is an “in class” and “closed-book” exam. Please do it completely on your own with no books or notes.

- (30 points) **Define** any three of the following terms. In each case, briefly provide an appropriate context for your definition.
 - A family of distributions $\mathcal{P} = \{P_\theta : \theta \in \Theta \subset \mathbb{R}\}$ (with densities $p_\theta = dP_\theta/d\mu$ with respect to a dominating measure μ) with *monotone likelihood ratio*.
 - An *unbiased* level α test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.
 - A similar test of $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$.
 - A *minimax decision rule* in a general decision problem with loss function $L(\theta, d)$.
- (30 points) **State** any three of the following results:
 - The Neyman - Pearson lemma.
 - The Karlin-Rubin theorem.
 - A theorem relating similar tests to tests with Neyman structure.
 - A theorem about admissibility properties of the sample mean \bar{X} when sampling from a normal distribution on \mathbb{R} and a contrasting theorem for sampling from a normal distribution on \mathbb{R}^d .
 - A theorem relating Bayes rules to minimax rules and least favorable prior distributions.
 - A conditional limit theorem about the large sample behavior of posterior distributions

Everyone should do problems 3.

- (36 points) A random variable X takes on the values 1, 2, 3, 4 with probability distribution $p_0(x)$ or $p_1(x)$ as follows:

x	1	2	3	4
$p_0(x)$.05	.07	.40	.48
$p_1(x)$.15	.35	.35	.15

- (a) For the usual 0 – 1 loss, find a most powerful test of size .10 for testing $H : p = p_0$ versus $K : p = p_1$ and determine its power.
- (b) Find a test ϕ which minimizes the sum of risks $E_0\phi + E_1(1 - \phi)$. What is the relationship between the minimized sum of risks and the total variation distance between P_1 and P_2 and why does this make intuitive sense?
- (c) If the losses are $L(1, 1) = L(0, 0) = 0$, $L(0, 1) = 3$, $L(1, 0) = 2$, and the prior is $\lambda = (\lambda_0, \lambda_1) = (.4, .6)$, find the Bayes rule d_B and the minimax rule d_M .

Do any two of problems 4 - 6.

- 4. (40 points) State and prove the “short form” of the generalized Neyman-Pearson lemma.
- 5. (40 points) Prove the following theorem (in the context of finitely many states of nature): Suppose that d_0 is Bayes with respect to $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\lambda_i > 0$ for $i = 1, \dots, l$. Then d_0 is admissible.
- 6. (40 points) In the context of X_1, \dots, X_n i.i.d. either P or Q with densities p and q with respect to some dominating measure μ , show that Neyman - Pearson tests are both size and power consistent under a simple restriction on the constants specifying the tests.

Do either problem 7 or problem 8.

- 7. (36 points) Suppose that X_1, \dots, X_n are i.i.d. Poisson(θ) random variables so that $P(X_1 = x) = \exp(-\theta)\theta^x/x!$ for $x \in \{0, 1, \dots\}$ and $\theta > 0$.

- (a) Find the MLE of θ .
- (b) If $\theta \sim \text{Gamma}(\alpha, \beta)$ so that

$$\lambda(\theta) = \frac{\beta^\alpha \theta^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{(0, \infty)}(\theta)$$

and $E(\theta) = \alpha/\beta$, find the posterior distribution of θ .

- (c) Find the Bayes estimator of θ for squared error loss. Is it consistent?
- (d) What is the asymptotic behavior of the posterior distributions you found in (b) when appropriately centered and normalized?

8. (36 points) Suppose that X_1, \dots, X_m are i.i.d. Geometric(μ) and that Y_1, \dots, Y_n are i.i.d. Geometric(ν) and independent of the X_i 's. (Thus $P(X_i = x) = \mu(1 - \mu)^{x-1}$, $x \in \{1, 2, \dots\}$ and $P(Y_i = y) = \nu(1 - \nu)^{y-1}$ for $y \in \{1, 2, \dots\}$ where $0 \leq \mu, \nu \leq 1$. (Reminder: $E(X_1) = 1/\mu$, $Var(X_1) = (1 - \mu)/\mu^2$.)
- (a) Consider testing $H : \nu \leq \mu$ versus $K : \nu > \mu$. Find the joint density function of $(\underline{X}, \underline{Y})$ and show that it can be written in the form of an exponential family with parameter of interest θ which takes a fixed value θ_0 on the boundary Θ_B (with an associated statistic $U(\underline{X}, \underline{Y})$), and a nuisance parameter ξ (with an associated sufficient statistic $T(\underline{X}, \underline{Y})$).
- (b) Find the conditional distribution of $U(\underline{X}, \underline{Y})$ given the sufficient statistic $T(\underline{X}, \underline{Y})$ for Θ_B .
- (c) Find the UMPU test of H versus K , specifying the constants involved as precisely as possible.
- (d) Describe the Bayes test of H versus K for 0 – 1 loss assuming a prior distribution Λ of (μ, ν) . Can you describe the rejection region of this test when Λ is the product of two independent Beta priors (i.e. $\mu \sim \text{Beta}(\alpha, \beta)$ and $\nu \sim \text{Beta}(\gamma, \delta)$)?