Statistics 582, Problem Set 1

Wellner; 1/7/2009

Reading: Chapter 4, Sections 4-6; Lehmann and Casella, TPE, section 6.6, pages 469 - 483; Ferguson, ACLST, Chapter 16-17, pages 107-118.
Due: Wednesday, January 14, 2009.

1. Suppose that $(X,Y),(X_1,Y_1),\ldots,(X_n,Y_n)$ are i.i.d. with bivariate normal distribution $N_2(\mu,\Sigma)$ where $\mu \in R^2$ and

$$\Sigma = \begin{pmatrix} \sigma^2 & \sigma\tau\rho \\ \sigma\tau\rho & \tau^2 \end{pmatrix}$$

where $\sigma^2 > 0$, $\tau^2 > 0$, and $\rho \in (-1,1)$.
(a) If we assume that $\mu_1 = \mu_2 \equiv \theta$ and Σ is known, what is the MLE of θ?
(b) If we assume that μ is known and $\sigma^2 = \tau^2 \equiv \theta$, what is the MLE of θ?
(c) What is the asymptotic distribution of the estimator you found in (b)?
(d) Under the same assumption as in (b), what is the MLE of ρ?
(e) What is the asymptotic distribution of the estimator you found in (d)?

2. Problem 1, page 117, Ferguson, ACILST. What happens if $\Theta = [1,\infty)$ or $(0,\infty)$?

3. Consider the model introduced in Ferguson, ACILST, problem 17.2, page 117. Show that Theorem 4.3, page 28, of the Chapter 4 notes (or Theorem 17, Ferguson, ACILST, page 114) applies to the MLE of θ in this model.

4. Suppose that $X, X_1,\ldots,X_n$ are i.i.d. Weibull(α_0,β_0) (if X has the Weibull(θ) distribution where $\theta = (\alpha,\beta)$, then $1 - F_\theta(x) = P_\theta(X > x) = \exp(-(x/\alpha)^\beta)$ for $x \ge 0$). Recall that the MLE $\widehat{\alpha}$ of α is given by

$$\widehat{\alpha} = \left\{ \frac{1}{n} \sum_{i=1}^n X_i^{\widehat{\beta}} \right\}^{1/\widehat{\beta}}$$

where $\widehat{\beta}$ is the MLE of β. As a simpler alternative to maximum likelihood, I propose to use the alternative estimator $\overline{\beta}_n$ of β obtained from the slope of an ordinary least squares fit of a Weibull Q-Q plot, and then estimate α by

$$\overline{\alpha}_n = \left\{ \frac{1}{n} \sum_{i=1}^n X_i^{\overline{\beta}_n} \right\}^{1/\overline{\beta}_n} .$$

(a) Suppose that $\overline{\beta}_n \to_p \beta_0$ is known. Show that $\overline{\alpha}_n \to_p \alpha_0$. [Hint: use a uniform strong law of large numbers.]
(b) Show that $\overline{\alpha}_n$ is a "pseudo-MLE" in the sense that $\overline{\alpha}_n$ maximizes $l_n(\alpha, \overline{\beta}_n)$.

5. (a) Suppose that $X_1,\ldots,X_n$ are i.i.d. with distribution P on R. Consider generalizing the result of the handout in class on 1/7/2009: if $V_n(r)$ is defined for $1 \le r \le 2$ by

$$V_n(r) \equiv \frac{1}{n} \sum_{i=1}^n |X_i - \overline{X}_n|^r .$$

If $E|X|^r < \infty$, show that

$$V_n(r) \rightarrow_{a.s.} v(r)$$

where $v(r) \equiv E|X_1 - \mu|^r$.

(b) Now suppose we generalize the problem considered in (a) by considering $X_1, \ldots, X_n$ i.i.d. P on R^d. Let $\| \cdot \|$ be the usual Euclidean metric in R^d, and consider

$$V_n(r) \equiv \frac{1}{n} \sum_{i=1}^{n} \|X_i - \overline{X}_n\|^r$$

for $1 \le r \le 2$ where $\overline{X}_n$ is the (multivariate) sample mean of the X_i's. Can the same method be used to show that $V_n(r) \rightarrow_{a.s.} v(r)$ where $v(r) \equiv E\|X_1 - \mu\|^r$ (assuming that $E\|X_1\|^r < \infty$)?

6. **Optional bonus problem.** On pages 116-117 of ACILST, Ferguson (see also Ferguson, T. S. (1982). An inconsistent maximum likelihood estimate. *J. Amer. Statist. Assoc.* **77**, 831–834) shows that $\hat{\theta}_n \rightarrow_{a.s.} 1$ no matter what θ_0 is true if $\delta(\theta) \rightarrow 0$ "fast enough".
(a) Show that $\hat{\theta}_n \rightarrow_{a.s.} 1$ continues to hold if

$$\delta(\theta) = (1 - \theta) \exp(-(1 - \theta)^{-c} + 1)$$

with $c > 2$. (Ferguson shows that $c = 4$ works.)
(b) Show that when $c = 2$, Ferguson's argument yields

$$\sup_{0 \le \theta \le 1} n^{-1} \log L_n(\theta) \ge \frac{n-1}{n} \log(M_n/2) + \frac{1}{n} \log \frac{1 - M_n}{\delta(M_n)} \rightarrow_d D$$

where

$$P(D \le y) = \exp\left(-\frac{1}{2(y - \log 2)} \right), \quad y \ge \log(2)\,.$$

That is, $D \stackrel{d}{=} \log 2 + 1/(2E)$ where E is an Exponential(1) random variable.
(c) What hypothesis in Wald's consistency theorem is violated in this example?