**Statistics 582, Problem Set 1**

Wellner; 1/7/2009

**Reading:** Chapter 4, Sections 4-6; Lehmann and Casella, TPE, section 6.6, pages 469 - 483; Ferguson, ACLST, Chapter 16-17, pages 107-118.
**Due:** Wednesday, January 14, 2009.

1. Suppose that $(X, Y), (X_1, Y_1), \ldots, (X_n, Y_n)$ are i.i.d. with bivariate normal distribution $N_2(\mu, \Sigma)$ where $\mu \in R^2$ and

$$\Sigma = \begin{pmatrix} \sigma^2 & \sigma\tau\rho \\ \sigma\tau\rho & \tau^2 \end{pmatrix}$$

where $\sigma^2 > 0$, $\tau^2 > 0$, and $\rho \in (-1, 1)$.
(a) If we assume that $\mu_1 = \mu_2 \equiv \theta$ and $\Sigma$ is known, what is the MLE of $\theta$?
(b) If we assume that $\mu$ is known and $\sigma^2 = \tau^2 \equiv \theta$, what is the MLE of $\theta$?
(c) What is the asymptotic distribution of the estimator you found in (b)?
(d) Under the same assumption as in (b), what is the MLE of $\rho$?
(e) What is the asymptotic distribution of the estimator you found in (d)?

2. Problem 1, page 117, Ferguson, ACILST. What happens if $\Theta = [1, \infty)$ or $(0, \infty)$?

3. Consider the model introduced in Ferguson, ACILST, problem 17.2, page 117. Show that Theorem 4.3, page 28, of the Chapter 4 notes (or Theorem 17, Ferguson, ACILST, page 114) applies to the MLE of $\theta$ in this model.

4. Suppose that $X, X_1, \ldots, X_n$ are i.i.d. Weibull$(\alpha_0, \beta_0)$ (if $X$ has the Weibull$(\theta)$ distribution where $\theta = (\alpha, \beta)$, then $1 - F_\theta(x) = P_\theta(X > x) = \exp(-(x/\alpha)^\beta)$ for $x \geq 0$). Recall that the MLE $\widehat{\alpha}$ of $\alpha$ is given by

$$\widehat{\alpha} = \left\{ \frac{1}{n} \sum_{i=1}^n X_i^{\widehat{\beta}} \right\}^{1/\widehat{\beta}}$$

where $\widehat{\beta}$ is the MLE of $\beta$. As a simpler alternative to maximum likelihood, I propose to use the alternative estimator $\overline{\beta}_n$ of $\beta$ obtained from the slope of an ordinary least squares fit of a Weibull Q-Q plot, and then estimate $\alpha$ by

$$\overline{\alpha}_n = \left\{ \frac{1}{n} \sum_{i=1}^n X_i^{\overline{\beta}_n} \right\}^{1/\overline{\beta}_n} .$$

(a) Suppose that $\overline{\beta}_n \rightarrow_p \beta_0$ is known. Show that $\overline{\alpha}_n \rightarrow_p \alpha_0$. [Hint: use a uniform strong law of large numbers.]
(b) Show that $\overline{\alpha}_n$ is a "pseudo-MLE" in the sense that $\overline{\alpha}_n$ maximizes $l_n(\alpha, \overline{\beta}_n)$.

5. (a) Suppose that $X_1, \ldots, X_n$ are i.i.d. with distribution $P$ on $R$. Consider generalizing the result of the handout in class on 1/7/2009: if $V_n(r)$ is defined for $1 \leq r \leq 2$ by

$$V_n(r) \equiv \frac{1}{n} \sum_{i=1}^n |X_i - \overline{X}_n|^r .$$

If $E|X|^r < \infty$, show that

$$V_n(r) \rightarrow_{a.s.} v(r)$$

where $v(r) \equiv E|X_1 - \mu|^r$.

(b) Now suppose we generalize the problem considered in (a) by considering $X_1, \ldots, X_n$ i.i.d. $P$ on $R^d$. Let $\|\cdot\|$ be the usual Euclidean metric in $R^d$, and consider

$$V_n(r) \equiv \frac{1}{n}\sum_{i=1}^{n} \|X_i - \overline{X}_n\|^r$$

for $1 \le r \le 2$ where $\overline{X}_n$ is the (multivariate) sample mean of the $X_i$'s. Can the same method be used to show that $V_n(r) \rightarrow_{a.s.} v(r)$ where $v(r) \equiv E\|X_1 - \mu\|^r$ (assuming that $E\|X_1\|^r < \infty$)?

6. **Optional bonus problem.** On pages 116-117 of ACILST, Ferguson (see also Ferguson, T. S. (1982). An inconsistent maximum likelihood estimate. *J. Amer. Statist. Assoc.* **77**, 831–834) shows that $\hat{\theta}_n \rightarrow_{a.s.} 1$ no matter what $\theta_0$ is true if $\delta(\theta) \rightarrow 0$ "fast enough".

(a) Show that $\hat{\theta}_n \rightarrow_{a.s.} 1$ continues to hold if

$$\delta(\theta) = (1-\theta)\exp(-(1-\theta)^{-c} + 1)$$

with $c > 2$. (Ferguson shows that $c = 4$ works.)

(b) Show that when $c = 2$, Ferguson's argument yields

$$\sup_{0 \le \theta \le 1} n^{-1} \log L_n(\theta) \ge \frac{n-1}{n}\log(M_n/2) + \frac{1}{n}\log\frac{1 - M_n}{\delta(M_n)} \rightarrow_d D$$

where

$$P(D \le y) = \exp\left(-\frac{1}{2(y - \log 2)}\right), \quad y \ge \log(2)\,.$$

That is, $D \stackrel{d}{=} \log 2 + 1/(2E)$ where $E$ is an Exponential(1) random variable.

(c) What hypothesis in Wald's consistency theorem is violated in this example?