Statistics 582, Problem Set 4

Wellner; 1/28/2009

Reading: Chapter 5, sections 1-6.
Due: Wednesday, February 4, 2009.
Reminder: Make up lectures, 11:30 - 12:20, Friday, 30 January and Monday, 2 February (in Thomson 125).

1. Let $\Theta = \{0, 1\} = \mathbf{A}$ where $0 =$ a patient has tuberculosis, $1 =$ a patient does not have tuberculosis. Let X be the number of positive reactions to two different tuberculosis tests, so that $\mathbf{X} = \{0, 1, 2\}$, and suppose that X has the following distributions

x	0	1	2
$p_0(x)$	.04	.16	.80
$p_1(x)$	.75	.20	.05

If the losses are given by $L(1, 1) = L(0, 0) = 0$, $L(0, 1) = 100$, $L(1, 0) = 10$, and the prior $\lambda = (\lambda_0, \lambda_1) = (.3, .7)$, find the Bayes rule d_B and the minimax rule d_M. Plot the risk set and label the non-randomized decision rules.

2. Let X be a random variable with finite first moment: $E|X| < \infty$. Show that $f(b) \equiv E|X - b|$ is minimized by $b =$ any median of the distribution F of X. [A median m of F is any value satisfying $F(m) = P(X \le m) \ge 1/2$ and $1 - F(m-) = P(X \ge m) \ge 1/2$; see Lehmann and Casella, TPE, page 62, problems 1.7 and 1.8.]

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. Exponential(θ) (so the X's have density $p_\theta(x) = \theta e^{-\theta x} 1_{(0,\infty)}(x)$. with respect to Lebesgue measure on R, and that $\theta \sim \Gamma(\alpha, \beta)$:

$$\lambda(\theta) = \beta \frac{(\beta\theta)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta\theta) 1_{[0,\infty)}(\theta).$$

(a) Find the Bayes rule $d_B(\underline{X})$ for estimation of θ with squared error loss $L(\theta, a) = |\theta - a|^2$. Find the Bayes rule $d_{Bw}(\underline{X})$ for estimation of θ with weighted squared error loss $L(\theta, a) = (\theta - a)^2/\theta$. Is the maximum likelihood estimator among either of these families of Bayes estimators?
(b) Are the Bayes estimators d_B and d_{Bw} consistent? What are the limit distributions of d_B and d_{Bw}? Compare them with the maximum likelihood estimator.
(c) Suppose that instead of the Gamma prior distribution, θ has the Pareto(θ_0, α) distribution with density λ given by

$$\lambda(\theta) = (\frac{\alpha}{\theta_0})(\frac{\theta_0}{\theta})^{\alpha+1} 1_{(\theta_0,\infty)}(\theta);$$

here $E(\theta) = \frac{\alpha}{\alpha-1}\theta_0$ where $\alpha > 1$ and $\theta_0 > 0$ are known. What can you say about the Bayes estimator for squared error loss with this prior? For what values of θ_0 is the Bayes rule consistent?

4. Consider testing the simple hypothesis $H_0 :\ X \sim P_0$ versus the simple alternative $H_1 :\ X \sim P_1$. Let ϕ be a test of H_0 versus H_1, and let $a \equiv E_1(1-\phi)$, $b \equiv E_0\phi$.
   (a) Find a test ϕ which minimizes $a + Db$ where D is a fixed number.
   (b) When $D = 1$, relate the minimized total $a + b$ to the risk and to the total variation distance $d_{TV}(P_0, P_1)$ between P_0 and P_1 (or $\int p_0 \wedge p_1 d\mu$ for a dominating measure μ, e.g. $P_0 + P_1$).
   (c) Carry the computations of (b) through in the context of problem 1 when the losses are $L(0,0) = L(1,1) = 0$, $L(0,1) = 10 = L(1,0)$, and the prior distribution is $\lambda = (\lambda_0, \lambda_1) = (.5, .5)$.

5. **Optional bonus problem:** Suppose that $\Theta = \{\theta_1, \theta_2\}$, $\mathbf{A} = \{a_1, a_2, a_3, a_4\}$, and that the loss function $L(\theta, a)$ is given by the following table:

θ/a	a_1	a_2	a_3	a_4
θ_1	1	1	2	2
θ_2	0	1	0	1

   Further suppose that $P_{\theta_j}(X = 0) = 1$ for $j = 1, 2$.
   (a) Find the decision risk set $\mathcal{R}$.
   (b) Find the decision rules that are Bayes with respect to the prior distribution $\lambda = (1, 0)$.
   (c) Show that the rule d_0 for which $R(\theta_1, d_0) = 1$ and $R(\theta_2, d_0) = 1$ is Bayes with respect to $\lambda = (1, 0)$ and also minimax, but that it is not admissible.
   (d) Relate this example to our theorem about admissibility of Bayes rules.