Statistics 582, Problem Set 8

Wellner; 2/25/2009

Reading: Chapter 6, sections 6.1 and 6.2 (through page 19); Lehmann and Casella, sections 3.1 - 3.4, 3.6 - 3.7, and 3.9.
Due: Wednesday, March 4, 2009.

1. (a) Show that the logistic distribution with location parameter θ having density

$$p_\theta(x) = \frac{\exp(x-\theta)}{(1+\exp(x-\theta))^2} = \frac{1}{2(1+\cosh(x-\theta))}$$

   has monotone likelihood ratio.
   (b) Unfortunately the result of (a) does not carry over to a sample of size n. If $X_1, \ldots, X_n$ are i.i.d. P_θ with density p_θ as in (a), then there is no $T(\underline{X})$ for which the MLR property holds. Nevertheless we can look for locally best tests. Find the locally best test of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$. How would you carry out this test?

2. Continuation of problem 2, problem set 7:
   (a) For P_0 and P_1 as given in problem 1 of problem set # 7, compute $d_{TV}(P_0, P_1)$, $H(P_0, P_1)$, and the affinity $\rho(P_0, P_1) = \int \sqrt{p_0 p_1} d\mu$.
   (b) For the product laws P_{0n} and P_{1n} (corresponding to observation of $X_1, \ldots, X_n$ i.i.d. P_0 or P_1 respectively) compute $\rho(P_{0n}, P_{1n})$ and $H(P_{0n}, P_{1n})$ for $n = 20,\ 50,\ 100$.
   (c) What does this imply about the test, ϕ_n say, based on $X_1, \ldots, X_n$ which minimizes the sum of risks?

3. Consider the Locally Most Powerful test ϕ for testing $H : \theta \le 0 \equiv \theta_0$ versus $K : \theta > 0 = \theta_0$ in Example 6.1.5.
   (a) Suggest two different approximations to the power of this test, one for local alternatives (of the form $\theta_n = t/\sqrt{n}$ with $t > 0$), and the other for fixed alternatives, $\theta > 0$.
   (b) What is the behavior of each of these two approximations for large values of θ? Which of them shows that the power function decreases to 0 as $\theta \to \infty$? Why?

4. Let $X_1, \ldots, X_n$ be a sample of size n from the uniform distribution $U(0, \theta)$. Sufficiency reduces the problem to $T = \max X_i$.
   (a) Find the class of all Neyman-Pearson best tests of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

(b) Find the subclass of the tests that are independent of θ_1. These are UMP tests of H_0 versus $H_1' : \theta > \theta_0$.
(c) Show that the test $\phi(t) = 1\{t > \theta_0\} + \alpha 1\{t \leq \theta_0\}$ is UMP of size α for testing $H_0' : \theta \leq \theta_0$ versus $H_1' : \theta > \theta_0$ but that ϕ is not admissible.
(d) Show that $\phi(t) = 1\{[t > \theta_0] \cup [t \leq b]\}$ where $b = \theta_0 \alpha^{1/n}$ is a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $\theta \neq \theta_0$.

5. Let X and Y be random variables with joint density

$$p_{X,Y}(x, y) = \lambda\mu \exp(-\lambda x - \mu y) 1_{(0,\infty)}(x) 1_{(0,\infty)}(y).$$

(a) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda \leq \mu + 1$ versus $H_1 : \lambda > \mu + 1$.
(b) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda = \mu$ versus $H_1 : \lambda \neq \mu$.
(c) Find a UMP unbiased test of size $\alpha = .2$ for testing $H_0 : \lambda \geq 2\mu$ versus $H_1 : \lambda < 2\mu$.
(d) What happens when $X_1, \ldots, X_m$ are i.i.d. Exponential(λ) and $Y_1, \ldots, Y_n$ are i.i.d. Exponential(μ)?

6. **Optional bonus problem:** Show that the Cauchy scale family of distributions given by

$$p_\theta(x) = \frac{1}{\pi\theta} \frac{1}{1 + (x/\theta)^2}$$

does not have monotone likelihood ratio, but that the distribution of the sufficient statistic $|X|$ (where $X \sim P_\theta$ with Cauchy density p_θ as in the last display) does have monotone likelihood ratio.