Statistics 591B, Problem Set 3

Wellner; November 2, 2017

Due: Thursday, November 16, 2017

1. Let $h(x) \equiv x(\log x - 1) + 1$. Show that for $f \ge 0$

$$Ent_\mu(f) = \int h(f)d\mu - h\left(\int f d\mu\right).$$

2. Suppose that f satisfies the Gaussian log-Sobolev inequality:

$$Ent_\gamma(f^2) \le 2\int |\nabla f|^2 d\gamma.$$

Show that this implies that f satisfies a Gaussian Poincaré-type inequality; that is, for $f \in C^1$ and some constant C

$$Var(f(Z)) \le CE\{\|\nabla f(Z)\|^2\}.$$

Hint: Apply the log-Sobolev inequality to $(1 + \epsilon f)$ and expand the resulting functions of ϵ.

3. Use Hoeffding's Lemma (BLM, Lemma 2.2) to prove Pinsker's inequality

$$d_{TV}^2(Q, P) \le \frac{1}{2}K(Q, P).$$

See BLM, Theorem 4.19, p. 103.

4. BLM, page 155, problem 5.5: (a) Prove the following variant of Theorem 5.3. Let $f : \{-1, 1\}^n \to \mathbb{R}$ and let X be uniformly distributed on $\{-1, 1\}^n$. Let $\nu > 0$ satisfy

$$\sum_{i=1}^n (f(x) - f(\overline{x}^{(i)}))^2 \ \le \ \nu$$

for all $x \in \{-1, 1\}^n$. (Note that, as opposed to the statement of Theorem 5.3, the positive part is omitted in the definition of ν.) Prove that, for all $t > 0$, $Z = f(X)$ satisfies

$$P(Z - E(Z) > t) \le \exp(-2t^2/\nu). \qquad (1)$$

Hint: Proceed as in the proof of the theorem, but instead of using the simple convexity argument establish first that for real numbers $z \geq y$,

$$(e^{z/2} - e^{y/2})^2 \leq \frac{(z-y)^2}{8}(e^z + e^y). \tag{2}$$

Use this to show that

$$\text{Ent}(e^{\lambda f(X)}) \leq \frac{1}{2}\sum_{i=1}^{n}\left\{\left(e^{\lambda X/2} - e^{\lambda f(\overline{X}^{(i)})/2}\right)^2\right\} \leq E\left\{\frac{\nu\lambda^2}{8}e^{\lambda f(X)}\right\}$$

(b) Show that the inequality (1) contains Hoeffding's inequality with the right constant in the exponent for the special case of symmetric Bernoulli random variables: If $X_1, \ldots, X_n$ are independent Rademacher random variables, then

$$P(n^{-1/2}\sum_{1}^{n} X_i > t) \leq \exp(-t^2/2) \quad \text{for all} \quad t > 0.$$

(c) Show that the inequality (2) can be rewritten as a lower bound for the geometric mean $\sqrt{ab}$ of two positive numbers a, b, thereby complementing the usual arithmetic mean - geometric mean upper bound, $\sqrt{ab} \leq (a+b)/2$.

5. (a) Suppose that $X_1, \ldots, X_n$ are i.i.d. P on $\mathbb{R}$ with $E|X_1| < \infty$. Consider the measure of dispersion $D_n \equiv n^{-1}\sum_{i=1}^{n}|X_i - \overline{X}_n|$ Use a Glivenko-Cantelli theorem to show that $D_n \rightarrow_{a.s.} E|X_1 - \mu| \equiv d$ where $\mu = E(X_1)$.
(b) Now suppose that the X_i's and D_n are as defined in (a), but now assume that $E(X_1^2) < \infty$. Use a Donsker theorem to show that $\sqrt{n}(D_n - d) \rightarrow_d$ something and identify "something". *Hint:* This is from Pollard (1989), *Statistical Science* **4**, 341-366.