Statistics 593A, Problem Set 1

Wellner; 4/3/2014

Due: Thursday, April 17, 2014

1. BLM, page 46, problem 2.3: Prove the Chebychev-Cantelli inequality: for any real-valued random variable Y and $t > 0$

$$P(Y - E(Y) > t) \le \frac{Var(Y)}{Var(Y) + t^2}.$$

   Compare this to the usual Chebychev inequality.

2. BLM, page 47, problem 2.4: Prove the Paley-Zygmund inequality: If Y is a non-negative random variable, then for any $a \in (0, 1)$

$$P(Y \ge aE(Y)) \ge (1 - a)^2 \frac{(EY)^2}{E(Y^2)}.$$

3. BLM, page 47, problem 2.7: Prove that if Z is a centered normal random variable with variance σ^2, then

$$P(Z \ge t) \le \frac{1}{2} \exp\left(-\frac{t^2}{2\sigma^2}\right) \quad \text{for all} \quad t > 0.$$

4. BLM, page 47, problem 2.7: Elementary inequalities:

$$-\log(1-u) - u \le \frac{u^2}{2(1-u)} \quad \text{for} \quad u \in (0,1);$$

$$\overline{h}(u) = (1+u)\log(1+u) - u \ge \frac{u^2}{2(1+u/3)} \quad \text{for} \quad u > 0;$$

$$h_1(u) = 1 + u - \sqrt{1+2u} \ge \frac{u^2}{2(1+u)}, \quad \text{for} \quad u > 0.$$

5. Verify the elementary calculation required to show that

$$\psi^*(t) \equiv \sup_{0<\lambda<1/c} \left(t\lambda - \frac{\nu\lambda^2}{2(1-c\lambda)}\right) = \frac{\nu}{c^2} h_1\left(\frac{ct}{\nu}\right)$$

   where $h_1(u) = 1 + u - \sqrt{1+2u}$, $u > 0$.

6. **Bonus problem:** Apply Hoeffding's inequality, Bennett's inequality, and the Bernstein inequality when the X_i's are i.i.d. Bernoulli(p) random variables with $0 < p \leq 1/2$, and compare the resulting bounds with the exact bound derived for $P(S \geq t)$ in class on 4/1 and 4/3, namely $P(S \geq t) \leq \exp(-nh_p(p + t/n))$ where

$$h_p(a) = a \log \frac{a}{p} + (1-a) \log \frac{1-a}{1-p}$$

is the Kullback-Leibler divergence $K(\text{Bern}(a), \text{Bern}(p))$.