Statistics 593A, Problem Set 3

Wellner; 5/1/2014

Due: Thursday, May 15, 2014

1. BLM, page 155, problem 5.5: (a) Prove the following variant of Theorem 5.3. Let $f : \{-1,1\}^n \to \mathbb{R}$ and let X be uniformly distributed on $\{-1,1\}^n$. Let $\nu > 0$ satisfy

$$\sum_{i=1}^{n} (f(x) - f(\overline{x}^{(i)}))^2 \ \le \ \nu$$

for all $x \in \{-1,1\}^n$. (Note that, as opposed to the statement of Theorem 5.3, the positive part is omitted in the definition of ν.) Prove that, for all $t > 0$, $Z = f(X)$ satisfies

$$P(Z - E(Z) > t) \le \exp(-2t^2/\nu). \tag{1}$$

Hint: Proceed as in the proof of the theorem, but instead of using the simple convexity argument establish first that for real numbers $z \ge y$,

$$(e^{z/2} - e^{y/2})^2 \le \frac{(z-y)^2}{8}(e^z + e^y). \tag{2}$$

Use this to show that

$$\mathrm{Ent}(e^{\lambda f(X)}) \le \frac{1}{2}\sum_{i=1}^{n} \left\{ \left(e^{\lambda X/2} - e^{\lambda f(\overline{X}^{(i)})/2} \right)^2 \right\} \le E\left\{ \frac{\nu\lambda^2}{8} e^{\lambda f(X)} \right\}$$

(b) Show that the inequality (1) contains Hoeffding's inequality with the right constant in the exponent for the special case of symmetric Bernoulli random variables: If $X_1, \ldots, X_n$ are independent Rademacher random variables, then

$$P(n^{-1/2}\sum_{1}^{n} X_i > t) \le \exp(-t^2/2) \quad \text{for all} \ \ t > 0.$$

(c) Show that the inequality (2) can be rewritten as a lower bound for the geometric mean $\sqrt{ab}$ of two positive numbers a, b, thereby complementing the usual arithmetic mean - geometric mean upper bound, $\sqrt{ab} \le (a+b)/2$.

2. BLM, page 156, problem 5.8: (Littlewood's inequality for real Rademacher sums) Let $Z = |\sum_1^n b_i\epsilon_i|$ where $b_1, \ldots, b_n \in \mathbb{R}$ are fixed coefficients and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d Rademacher random variables. Show first by elementary arguments that $E[Z^4] \leq 3(E[Z^2])^2$. Next use Hölder's inequality to show that $E[Z^2] \leq (EZ)^{2/3}(E[Z^4])^{1/3}$. Conclude that $E[Z^2] \leq 3(EZ)^2$. Is the comparison between the fourth and second moment improvable?

3. Suppose that Z is as in the previous problem. Use an exponential bound for $P(Z > t)$ to show that for every $p \geq 1$ there exist positive constants A_p and B_p such that
$$A_p\{EZ^2\}^{1/2} \leq \{EZ^p\}^{1/p} \leq B_p\{EZ^2\}^{1/2}$$
where $EZ^2 = \sum_1^n b_i^2$. (These are known as Khinchine's inequalities.)
Hint: See Ledoux and Talagrand (1991), page 91.

4. BLM, page 157, problem 5.14: Provide details for the first step of the proof of Theorem 5.8: *Hint:* By total boundedness and sample path continuity, $Z = \sup_{t\in\mathcal{D}} X_t$ where $\mathcal{D}$ is a dense countable subset of $\mathcal{T}$. Use the Gaussian Poincaré inequality for finite subsets and monotone convergence to show that Z has an expected value (by relating it to the median of Z). Then use monotone convergence and the theorem for finite sets to finish the proof.
Hint: See van der Vaart & W (1996), pages 438-439.

5. **Bonus problem:** BLM, page 157, problem 5.16: (Adapting Herbst's argument) Let $X_1, \ldots, X_n$ be independent standard Gaussian random variables. Let f denote a differentiable function on $\mathbb{R}^n$ such that $E\{\exp(\lambda\|\nabla f(X_1, \ldots, X_n)\|^2)\} < \infty$ for $\lambda < \lambda_0$ where λ_0 may be ∞. Let $Z = f(X_1, \ldots, X_n)$. Prove that for λ, θ satisfying $\lambda/\theta < \lambda_0$ and $\lambda\theta < 2$,
$$\log E\{\exp(\lambda(Z - E(Z)))\} \leq \frac{\lambda\theta}{2(1 - \lambda\theta/2)} \log E\{\exp(\lambda\|\nabla f\|^2/\theta)\}.$$
Hint: Starting from the Gaussian logarithmic Sobolev inequality, use Corollary 4.15 to upper bound $E\{\|\nabla f\|^2 \exp(\lambda Z)\}$. Apply this result when f is the squared norm of the orthogonal projection of X on some linear subspace of $\mathbb{R}^n$.