BETA-PATHS IN THE HAMMERSLEY PROCESS

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Abstract. We study the asymptotics of β-paths in the Hammersley process with sources and sinks, of intensities λ and ρ respectively, introduced by Groeneboom (2002). We derive a strong law of large number for those paths in the regime $\lambda\rho \leq 1$ and we show that its fluctuation exponent is at most $2/3$. Examples of β-paths are the space-time path of a second-class particle in the Hammersley process and also the space-time path of the interface between two PNG droplets.

1. Introduction

Aldous and Diaconis [1] introduced a continuous time version of the interacting particle process in Hammersley [14] using the following rule. Start with a Poisson point process $\mathcal{P}$ of intensity 1 in the positive quadrant $\mathbb{R}^2$ and move the interval $[0, x]$ vertically through a realization of this point process; if this interval catches a point that is to the right of the points caught before, a new point (or particle) is created in $[0, x]$; otherwise we shift to this point the previously caught point that is immediately to the right and belongs to $[0, x]$. The number of particles, resulting from this rule, at time t on the the interval $[0, x]$ is denoted by $L(x, t)$ and the evolving particle process $(L(., t)\,, t \geq 0)$ is called the *Hammersley process*. In this work we consider a extension of the Hammersley process, as introduced by Groeneboom [13], where we also have, independently of $\mathcal{P}$, mutually independent Poisson point process on the x- and t-axis, say $\mathcal{X}$ and $\mathcal{T}$, with intensities $\lambda \geq 0$ and $\rho \geq 0$ respectively. Now we have the following rule: start the interacting particle process with a non-empty configuration of points, called sources, on the x-axis, which are subjected to the Hammersley interacting rule in the positive quadrant and which escape through the points on the t-axis, called sinks, if such a sink appears to the immediate left of a particle. We follow the terminology adopted by Groeneboom [12] and we call α-points the points of the process $\mathcal{P}$ and β-points the concave corners (or left turns) of the space-time paths of the particles in the Hammersley process with sinks and sources (Figure 1). The interest in β-points begins with the observation that $N(x, t)$ is equal to the number of α-points in $[0, x] \times [0, t]$ minus the number of β-points in $[0, x] \times [0, t]$. As showed by Cator and Groeneboom [6], we also have that this particle process, with Poisson sources of intensity λ and no sinks, behaves below an asymptotically linear wave of slope λ^2 through the β-points as a stationary process. They also proved, in the stationary regime $\rho = 1/\lambda$,

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a Burke theorem for the β-points showing that these points inherit the Poisson property from the α-points.

Here we study the asymptotics of what we call β-paths. To be more precise we first need to introduce some notation. For $P, Q \in \mathbb{R}_+^2$ define that $P \prec Q$ if both coordinates of P are lower or equal than those of Q. Denote by $\Delta_0, \Delta_1, \Delta_2 \dots$ the space-time paths of the particles in the Hammersley process with the convention that $\Delta_0 = \{(0,0)\}$ and that Δ_k lies below Δ_{k+1}. We say that $(P_n)_{n\geq 0}$, a sequence of points in $\mathbb{R}_+^2$, is a β-path if it satisfies: i) $P_n \prec P_{n+1}$; ii) $P_n \in \Delta_n$; iii) P_n is a β-point.

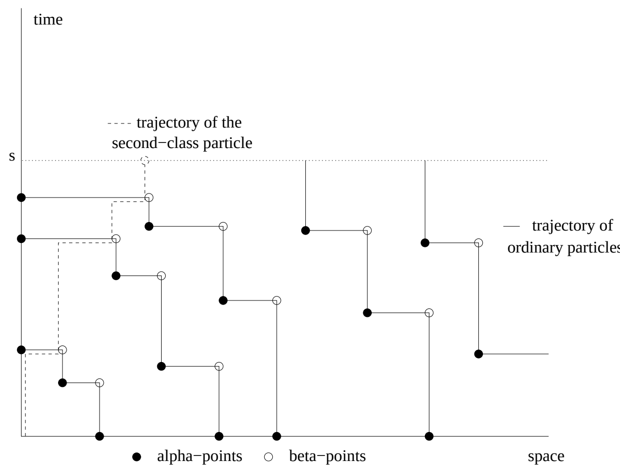

FIGURE 1. Second-class particle

1.1. **Second-class particle in the Hammersley process.** To give an example of a β-path in the Hammersley process with sinks (i.e. $\rho > 0$) we introduce the concept of a *second-class particle*. A second-class particle is a special particle that starts at the origin and jumps to the previous position of the ordinary particle that exits through the first sink at the time of the exit, and successively jumps to the previous position of particles directly to the right of it, at times where these particles jump to a position to the left of the second-class particle (Figure 1).

The position of the second-class particle at time t is denoted by X_t. Thus if τ_n denotes the the time of the n-th jump of the second-class particle (with the convention that $\tau_0 = 0$) then $(Q_n)_{n\geq 0}$, where $Q_n := (X_{\tau_n}, \tau_n)$, is a β-path.

The concept of a second class particle was first introduced by Ferrari [9] in the totally asymmetric exclusion process (TASEP) to identify the behavior of a perturbation of the particle system. He showed that, in the TASEP context, the trajectory of a second-class particle is asymptotically driven by the characteristics of the hydrodynamic equation associated to this particle system. A similar phenomenon has been observed by Cator and Groeneboom [6] in the stationary Hammersley process: with probability one, the space-time path of a second-class particle moves along the characteristic of the hydrodynamic equation associated to the Hammersley process

$$\partial_t u + \partial_x \left(-\frac{1}{u} \right) = 0 \quad \text{in} \quad (0,\infty)^2 \,, \tag{1.1}$$

with initial condition $u(x,0) = \lambda$ for $x \in (0,\infty)$ and $u(0,t) = \rho$ for $t \in (0,\infty)$, which in the stationary case has slope λ^2 [20]. We prove that[1]:

Theorem 1. *Let $(X_t, t \le 0)$ be the trajectory of a second class particle which is initially at the origin in the Hammersley process with sinks, i.e. $\rho > 0$, and $\lambda\rho < 1$. Then almost surely*

$$\exists \lim_{t\to\infty} \frac{X_t}{t} = Z$$

where Z is a random variable with the following distribution:

$$\mathbb{P}(Z \le r) = \begin{cases} 0, & r \le \rho^2, \\ \frac{\rho^{-1} - \sqrt{r^{-1}}}{\rho^{-1} - \lambda}, & \rho^2 < r \le \lambda^{-2}, \\ 1, & \lambda^{-2} < r. \end{cases}$$

Remark 1. *For $\lambda\rho = 1$ Cator and Groeneboom* [6] *proved that almost surely*

$$\exists \lim_{t\to\infty} \frac{X_t}{t} = \frac{1}{\lambda^2} \,.$$

If one looks to the characteristics of the hydrodynamic equation (1.1) then one can see that there is only one characteristic emanating from the origin when $\lambda\rho = 1$ and that there are infinitely many characteristics when $\lambda\rho < 1$, which corresponds to rarefaction front. Together with Remark 1, Theorem 1 shows that, analogously to the TASEP context, the second-class particle follows the characteristics when there is only one, i.e. $\rho\lambda = 1$, and that it chooses randomly one of the characteristics when $\lambda\rho < 1$ [2].

1.2. **The interface between two PNG droplets.** To give another example of a β-path we consider a layer by layer growth model [15], named the *Polynuclear growth* (PNG) model, in 1+1 dimensional which is closely related to the Hammersley process . The surface at time $s \ge 0$ is described by an integer-valued function $h(.,s) : z \in \mathbb{R} \to h(z,s) \in \mathbb{Z}$, named the height profile at time s, for which the discontinuity points have upper limits. We consider the initial condition $h(0,z) = 0$ for all $z \in \mathbb{R}$. For each $s > 0$ the function $h(.,s)$ has jumps of size one at the discontinuity points, called *up-step* if h increases and *down-step* if h decreases (Figure 2. A *nucleation* event at position (z,s) is a creation of a spike, a pair of up- and down-steps, over the previous layer. The up-steps move to the left with unit speed and the down-steps move to the right with unit speed. When a up- and down-steps collide they disappear.

[1]We remark that the limit in distribution of the second-class particle when $\lambda\rho < 1$ was also identified by Cator and Dobrynin [5].

[2]When $\lambda\rho > 1$ the characteristics of (1.1) emanating from $\{0\} \times (0,\infty)$ collide with the characteristics emanating from $(0,\infty) \times \{0\}$ giving rise to a shock. We do expect that, in this case, the second-class particle is driven by the shock to the limit value ρ/λ, with Gaussian fluctuations [9].

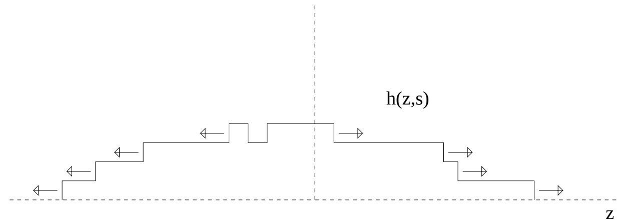

FIGURE 2. PNG droplet

The nucleation events form a locally finite point process in space-time and it is constructed as folllows. Consider the transformation $(x,t) \to (z,s)$ that rotates the (x,t)-plane by $\pi/4$ (in the canonical orientation of $\mathbb{S}^1$, the set of unit vectors). Then the random points in $\{z=-s\}$ will correspond to the sinks in t-axis, the random points in $\{z=s\}$ will correspond to the sources in x-axis and finally, the random points in $\{|z|<s\}$ will correspond to the α-points on the strictly positive quadrant $\mathbb{R}^2$. We assume that outside $\{|z|\geq s\}$ there is no nucleation event. To know the value of $h(z,s)$ one draw the trajectories of the up- and down-steps in the space-time (z,s)-plane. When two of these paths meet (as s increases) they stop, which reflects the disappearing of the corresponding up- and down-step. In this way the space-time is divided into regions bounded by piecewise straight lines with slopes equal to 1 or -1 (Figure 3). For fixed $z\in\mathbb{R}^2$ the height $h(z,s)$ is constant in each region. One can also see that the rotated trajectories of the (ordinary) particles will correspond to the lines defined by the trajectories of the up- and down-steps and thus we will have that $h(z,s)=L(x,t)$. Notice also that the rotated β-points will corresponds to the collisions between up- and down- steps (Prahofer and Spohn [18]).

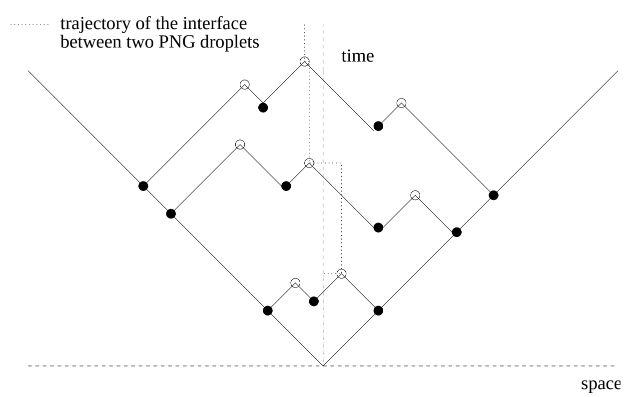

FIGURE 3. The nucleation events

Now assume that the initial layer is divided into two different types of materials, say 1 if $z<0$ and 2 if $z>0$. Consider the rule which stipulates that if a spike is created over material j then it will belong to this material, and that when a down-step of type 1 collide with a up-step of type 2 they stop (Figure 4). Thus an interface between these two growing materials is formed and we denote by φ_n the position of this interface at the

n-th layer, and by σ_n the time for which the down-step of type 1 collide with the up-step of type 2 at layer n-th layer, with the convention that $\varphi_0 = 0$ and $\sigma_0 = 0$. By the above remarks, one can see that the path $(R_n)_{n\geq 0}$, where $R_n := (\varphi_n, \sigma_n)$, is a rotated β-path (see the trajectory in Figure 3).

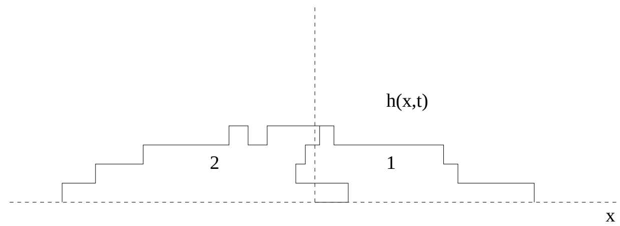

FIGURE 4. Two PNG droplets

In the literature, the interest on these two-type growth models begins with Derrida and Dickman [8] who performed numerical simulations to predict the value of the roughening exponent of the interface between two growing Eden species (first-passage percolation growth) in $\mathbb{Z}^2$ starting from a deterministic corner of angle α (see also Saito and Muller-Krumbhaar [19]). They found a quite different qualitative behavior of this interface when the angle goes from $\alpha < \pi$ to $\alpha > \pi$, which indicates that its asymptotic inclination goes from deterministic to random[3]. Later Ferrari, Martin and Pimentel [11] introduced a last-passage competing growth model in $\mathbb{Z}^2$, starting from random corner of angle α, and proved that the resulting competition interface, i.e. the interface between the two competing materials, can be linearly map into the trajectory of the second class particle in the TASEP [10]. This map allowed them to give an explicit description of the asymptotic behavior of the competition interface as a function of the angle α, and to understand the transition from $\alpha < \pi$ to $\alpha > \pi$ in terms of the characteristics of the associated hydrodynamic equation. In the two-type PNG model we show that something similar happens. Precisely, define the process $(\varphi(s),\ s \geq 0)$ by setting $\varphi(s) = \varphi_n$ for $s \in [\sigma_n, \sigma_{n+1})$. We show that:

Theorem 2. *Let $(\varphi(s)\, s \geq 0)$ be the trajectory of the competition interface in the two type PNG model with $\lambda\rho \leq 1$. Then almost surely*

$$\exists \lim_{s\to\infty} \frac{\varphi_s}{s} = \begin{cases} \frac{1-\lambda^2}{1+\lambda^2}, & \text{if } \lambda\rho = 1, \\ W, & \text{if } \lambda\rho < 1. \end{cases}$$

where $W = W(\mathcal{P}, \mathcal{X}, \mathcal{T})$ is a non trivial random variable.

We note that, differently from the TASEP×LPP set-up studied by Ferrari, Martin and Pimentel, in the Hammersley×PNG context we do not have a direct map between the second class particle and the competition interface. By this reason we are not able, in principle, to use hydrodynamic methods to calculate the law of the limit inclination W.

[3]For more on multi-type growth models in the first-passage percolation context see Pimentel [17] and the references therein.

1.3. **Convergence of beta-paths and its fluctuations.** Motivated by the above results concerning these two examples we study the asymptotics of an arbitrary β-path and we prove the following. Denote by $ang(P,Q)$ the angle in $[0,2\pi)$ between P and Q.

Theorem 3. *Assume that $(P_n)_{n\geq 1}$ is a β-path in the Hammersley process with $\lambda\rho\leq 1$. Then almost surely*

(1) $\lim_{n\to\infty}\frac{P_n}{|P_n|}=V=(\cos\theta,\sin\theta)$, *where $\theta\in[0,\pi/2]$ is given by*

$$\cot(\theta)=\begin{cases}\frac{1}{\lambda^2}, & \text{if } \lambda\rho=1,\\ U, & \text{if } \lambda\rho<1.\end{cases}$$

and $U=U(\mathcal{P},\mathcal{X},\mathcal{T})$ is a non trivial random variable.

(2) *Further, if $\lambda\rho<1$ then for all $\delta>0$ almost surely there exists a constant $c>0$ so that $ang\,(P_n,V)\leq c|P_n|^{-\delta}$ for all large n.*

We remark that the second item in Theorem 3 tell us that, in the rarefaction regime $\lambda\rho<1$, for all $\epsilon>0$ the fluctuations of a β-path $(P_n)_{n\geq 0}$ around its asymptotic inclination are at most of order $|P_n|^{2/3+\epsilon}$. We do believe that Theorem 3 is almost optimal, i.e. that the correct exponent should be 2/3. In the stationary regime $\lambda\rho=1$, Cator and Groeneboom [7] proved that indeed the right exponent for the fluctuations of a second-class particle in the Hammersley process is 2/3, and this indicates that the same should be true for β-paths (in particular, also for the interface between two stationary PNG droplets).

2. Proofs of the main results

2.1. **Convergence in distribution.** The limit law of the second class particle follows from the computation below as well as from Cator and Dobrynin [5]. Let $\eta_t, t\geq 0$ be the point process obtained by starting with a Poisson Process of intensity λ in $(0,\infty)$ at time 0, and letting it develop according to Hammersley process on $(0,\infty)$, with Poisson sinks of intensity ρ with $\lambda\rho<1$ and a Poisson point process of intensity 1 in the interior of the first quadrant. Furthermore, let $\eta^h_t, t\geq 0$ be the process coupled to $\eta_t, t\geq 0$, by using the same points in the first quadrant and in the y-axis as used for η. At time 0, we consider the same sources on the interval (h,∞) and on the interval $[0,h]$ we add an independent Poisson process of intensity $\rho^{-1}-\lambda$. Denote by $\eta_t[x,y]$ the number of particles on the interval $[x,y]$ at time t and similarly by $\eta^h_t[x,y]$ for the coupled process.

Let $X^\epsilon_t=\epsilon X_{t\epsilon^{-1}}$,

$$F^\eta_h(r,t)=\eta_t[0,r]-\eta^h_t[0,r]$$

and

$$F^\eta_{h,\epsilon}(r,t)=F^\eta_h(r\epsilon^{-1},t\epsilon^{-1})\,.$$

By coupling, we have that

$$\mathbb{E}\left(F^\eta_{h,\epsilon}(r,t)\right)=-\left(\rho^{-1}-\lambda\right)he^{-\rho^{-1}h}\mathbb{P}\left(X^h_{t\epsilon^{-1}}\leq r\epsilon^{-1}\right)+o(h)$$

Dividing by h and taking limit when h and ϵ go to 0 we have that

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\mathbb{E}\left(\frac{F^{\eta}_{h,\epsilon}(r,t)}{h}\right) = -\left(\rho^{-1}-\lambda\right)\lim_{\epsilon\to 0}\mathbb{P}\left(X_{t\epsilon^{-1}}\leq r\epsilon^{-1}\right)$$

On the other hand,

$$\begin{aligned}\mathbb{E}\left(F^{\eta}_{h,\epsilon}(r,t)\right) =\\ \mathbb{E}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h,r\epsilon^{-1}]-\eta^h_0[0,h]\right) =\\ \mathbb{P}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h,r\epsilon^{-1}]=1,\eta^h_0[0,h]=0\right) -\\ \mathbb{P}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h,r\epsilon^{-1}]=0,\eta^h_0[0,h]=1\right)+o(h)\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\mathbb{P}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h,r\epsilon^{-1}]=1,\eta^h_0[0,h]=0\right) -\\ \mathbb{P}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h,r\epsilon^{-1}]=0,\eta^h_0[0,h]=1\right) =\\ \mathbb{P}\left(\eta_{t\epsilon^{-1}}[r\epsilon^{-1}-h,r\epsilon^{-1}]=1\right)-\mathbb{P}\left(\eta^h_0[0,h]=1\right)+o(h)\end{aligned} \tag{2.3}$$

Since

$$\mathbb{P}\left(\eta^h_0[0,h]=1\right) = (\rho^{-1}h)e^{-\rho^{-1}h}$$

we get that

$$\lim_{\epsilon\to 0}\lim_{h\to 0}\mathbb{E}\left(\frac{F^{\eta}_{h,\epsilon}(r,t)}{h}\right) = u(r,t)-\rho^{-1}$$

and

$$\lim_{\epsilon\to 0}\mathbb{P}\left(X^{\epsilon}_t\leq r\right) = \frac{\rho^{-1}-u(r,t)}{\rho^{-1}-\lambda} \tag{2.4}$$

where $u(r,t)$ is the unique entropic solution of (1.1), which is given by

$$\begin{aligned}u(x,t) &= \rho^{-1}\ 1\{\rho^{-2}x\leq t\}+\sqrt{tx^{-1}}\ 1\{\lambda^2 x\leq t<\rho^{-2}x\}\\ &+\lambda\ 1\{t<\lambda^2 x\}\end{aligned} \tag{2.5}$$

(see [20]).

2.2. **Dual second class particle.** Now we define the concept of a *dual second-class particle*, introduced by Cator and Groeneboom [6]. To do so, recall that to determine the process $t\to L(.,t)$ at point x we shift until time t the interval $[0,x]$ vertically through a realization and we follow the Hammersley interacting rule allowing particles to escape through the sinks. By symmetry, we can also introduce the dual process $x\to L^*(x,.)$ by running the same rule, but now from left to right, i.e. sinks for L becomes sources for L^* and sources for L becomes sinks for L^*. Notice that, in the stationary regime $\lambda\rho=1$, both processes L and L^* have the same law. We denote X^* the second-class particle with respect to the dual process L^* and we denote by X^*_t the intersection between the space-time path of the dual second-class particle with $[0,\infty)\times\{t\}$. Trajectories of X and X^* are shown in Figure 5.

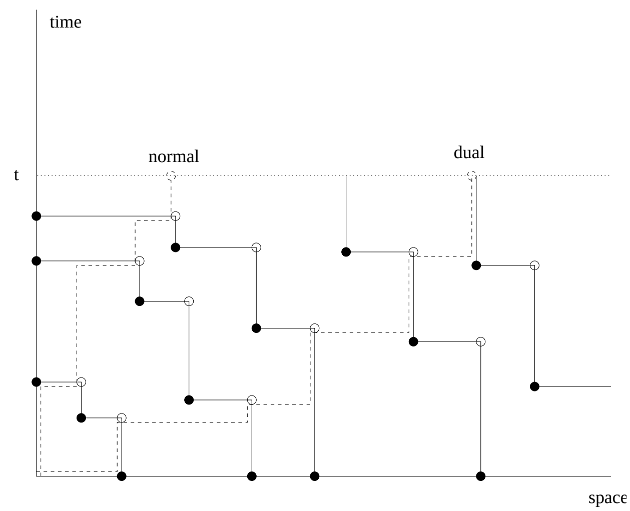

FIGURE 5. Normal and dual second-class particles

Remark 2. *From Remark 1 we have that if $\lambda\rho = 1$ then almost surely*

$$\exists \lim_{t\to\infty} \frac{X_t^*}{t} = \frac{1}{\lambda^2}\,.$$

From the convergence in distribution of X_t, we also have that X_t^*/t converges in distribution to Z' where

$$\mathbb{P}(Z'\leq r) = \begin{cases} 0, & r\leq \rho^2,\\ \frac{\sqrt{r}-\rho}{\lambda^{-1}-\rho}, & \rho^2 < r \leq \lambda^{-2},\\ 1, & \lambda^{-2} < r. \end{cases}$$

An easy but useful observation is that the β-paths $(Q_n)_{\geq 1}$ and $(Q_n^*)_{\geq 1}$, which corresponds to the normal and dual second-class particles are the left- and right-most β-paths, respectively:

Lemma 1. *Let $(Q_n)_{\geq 1}$ and $(Q_n^*)_{\geq 1}$ be the β-paths that corresponds to the normal and dual second-class particles respectively, and let $(P_n)_{\geq 1}$ be a β-path. Then*

$$\frac{Q_n(1)}{Q_n(2)} \leq \frac{P_n(1)}{P_n(2)} \leq \frac{Q_n^*(1)}{Q_n^*(2)}\,.$$

Remark 3. *Together with Lemma 1, the convergence in the regime $\lambda\rho = 1$ implies the convergence of β-paths in the same regime.*

Combining the convergence in distribution to a continuous random variable with Lemma 1 one also gets that:

Lemma 2. *If $\lambda\rho < 1$ then almost surely for all β-path $(P_n)_{n\geq 0}$ there exists a sufficiently small (random) $\epsilon > 0$ such that*

$$\rho^2 + \epsilon < \frac{P_n(1)}{P_n(2)} < \lambda^{-2} - \epsilon\,,$$

for all large n.

2.3. **Last-passage percolation.** To study the almost surely behavior of β-paths in the regime $\lambda\rho < 1$ we use some results concerning a Poisson last-passage percolation model defined as follows. For a given realization of the Poissonian points $\mathcal{P}, \mathcal{X}$ and $\mathcal{T}$, a weakly up/right path $(P_1, \ldots, P_l)$, starting at P and ending at Q, is a oriented and piecewise linear path γ connecting $P \prec P_1 \prec ... \prec P_l \prec Q$, where P_j are Poissonian points. The length $l(\gamma)$ of the path is the number of Poissonian points used by γ. Let $\Gamma(0, Q)$ be the set of weakly up/right paths from the origin to Q. The maximal length between 0 and Q is defined by

$$L_m(Q) = \max_{\gamma\in\Gamma(0,Q)} l\,(\gamma)\ . \tag{2.6}$$

Every $\gamma \in \Gamma(0, Q)$ such that $l(\gamma) = L_m(Q)$ is called a geodesic. We note that if $Q = (x, t)$ then $L_m(Q) = L(x, t)$. Indeed a longest weakly up/right path has to pick a point from each space-time crossing of the rectangle $[0, x] \times [0, t]$. Since the number of particle is also equal to the number of crossing paths on this rectangle, the equality $L_m = L$ follows. We will denote this common value by $L^\rho_\lambda(x, t)$ where λ and ρ are the intensities of the Poisson processes of sources and sinks respectively. When $x = t$ we simply write $L^\rho_\lambda(t)$. The last observation also brings that if we denote $G_k = \{L_m(Q) \leq k - 1\}$, then ∂G_k (the right-hand boundary of G_k) is equal to Δ_k, the trajectory of the kth particle in the Hammersley process.

The asymptotics of L were studied by Baik and Rains [2] and it follows from their work that if $\alpha_1, \alpha_2 \in [0, 1)$ then there exists constants $c_j > 0$ so that if $x \leq -c_1$ and $t \geq c_2$ then

$$\mathbb{P}(L^{\alpha_2}_{\alpha_1}(t) - 2t \leq xt^{1/3}) \leq c_3 e^{-|x|^3} \tag{2.7}$$

and that if $x \geq c_1$ and $t \geq c_2$ then

$$\mathbb{P}(L^{\alpha_2}_{\alpha_1}(t) - 2t \geq xt^{1/3}) \leq c_4 e^{-c_5 x^{3/2}}\ . \tag{2.8}$$

(see equations (5.2) and (5.14) in [2], Proposition 5.6 in [3], and [4]). Notice that for $\lambda\rho < 1$ and $\rho^2 < x/t < 1/\lambda^2$,

$$\mathbb{P}(L^\rho_\lambda(x, t) - 2\sqrt{xt} > k(\sqrt{xt})^{1/3}) = \mathbb{P}(L^{\alpha_2}_{\alpha_1}(\sqrt{xt}) - 2\sqrt{xt} > k(\sqrt{xt})^{1/3})$$

where $\alpha_1 = \lambda\sqrt{x/t}, \alpha_2 = \rho\sqrt{t/x} \in [0, 1)$.

With this in hand one can shows that, in the rarefaction region[4] $\{\rho^2 < x/t < 1/\lambda^2\}$, for all $\delta \in (0, 1/3)$ if a geodesic $\pi(0, P)$ walks to a remote site Q then it has to stay, after passing through P, in the cone with axis through 0 and P and angle of order $|P|^{-\delta}$, which

[4]The rarefaction region corresponds to the curved piece of the limit shape associated to last passage percolation model.

is the so called δ-straightness of geodesics introduced by Newman [16] in the first-passage percolation context. Precisely, for $\theta \in (0, \pi/4)$ and $P \in \mathbb{R}^2$ let $Co(P, \theta)$ be the cone with axis through P and 0 and with angle θ. For oriented path γ with $P \in \gamma$ we define

$$\gamma^{out}(P) = \{Q \in \gamma \, : \, P \prec Q\} \, .$$

Further, for $\alpha < \beta$, $R(\alpha, \beta)$ is the region in the positive quadrant between the lines $\{t = \alpha x\}$ and $\{t = \beta x\}$ (for $\beta = \infty$ we set $\{x = 0\}$). Then we have the following δ-straightness result:

Lemma 3. *Let $\delta \in (0, 1/3)$ and $\epsilon > 0$. Then almost surely there exists $M > 0$ with: for $Q \in \mathbb{R}^2_+$, $\gamma \in \Gamma(0, Q)$ and $P \in R(\rho^2 + \epsilon, 1/\lambda^2 - \epsilon)$ with $|P| > M$ and $P \in \gamma$ we have*

$$\gamma^{out}(P) \in Co(P, \frac{2|P|^{-\delta}}{1 - 2^{-\delta}}) \, .$$

For $\lambda = \rho = 0$ this is exactly Lemma 2.4 of Wuthrich [21]. To avoid repetition we omit the proof of Lemma 3, which follows by combining equations (2.7) and (2.8) with some geometrical ideas in Newman (1995).

The idea behind the proof of Theorem 3 is to show that for every $n \geq 1$ we can construct two geodesics, both starting from $(0, 0)$ and ending at P_n, such that the path $(P_0, \ldots, P_n)$ is enclosed by them (see Figure 6). Since geodesics are δ-straight, this will imply δ-straightness of β-paths.

Lemma 4. *Almost surely, if $(P_n)_{n \geq 1}$ is a β-path then for all $n \geq 1$ there exists two geodesics γ_n^+ and γ_n^- in $\Gamma(0, P_n)$ such that the oriented path $(P_0, \ldots, P_n)$ is enclosed by them.*

Proof of Lemma 4. Let $G_n^+ = (G_n^+(1), P_n(2))$ be the Poissonian point (coming from one of the three Poisson point processes) that first appear to the left of P_n in level ∂G_n. Fix $Q \in \mathbb{R}^2_+$ and let $A_Q = \{P \prec Q\}$. Suppose that $G_n^+, \ldots, G_{n-k}^+$ have already been defined for $k < n$. Then set $G_{n-(k+1)}^+$ to be the first Poissonian point in level $\partial G_{n-(k+1)} \cap A_{G_{n-k}^+}$ to the left of $P_{n-(k+1)}$. Notice that if one of the G_k^+ belongs to the t-axis then $G_0^+, ..., G_{k-1}^+$ belong too. By construction, the oriented path $(G_0^+, \ldots, G_n^+, P_n)$ is a geodesic (since it picks one point in each level behind P_n) which is always above $(P_0, \ldots, P_n)$. Similarly, we can construct a geodesic $(G_1^-, \ldots, G_n^-, P_n)$ which is above $(P_1, \ldots, P_n)$. In this case, we proceed as follows: let $G_n^- = (P_n(1), G^-(2)_n)$ be the Poissonian point that first appear to the right of P_n in level ∂G_n. Suppose that $G_n^-, \ldots, G_{n-k}^-$ have already been defined for $k < n$. Then set $G_{n-(k+1)}^-$ to be the first Poissonian point in level $\partial G_{n-(k+1)} \cap A_{G_{n-k}^-}$ to the right of $P_{n-(k+1)}$. Notice that if one of the G_k^- belongs to the x-axis then $G_0^-, ..., G_{k-1}^-$ belong too. By construction, the path $(G_0^-, \ldots, G_n^-, P_n)$ is a geodesic which is always below $(P_0, \ldots, P_n)$. $\square$

Proof of Theorem 3. By Lemma 2, almost surely for all β-path $(P_n)_{n \geq 1}$ there exists $\epsilon > 0$ so that $P_n \in R(\rho^2 + \epsilon, 1/\lambda^2 - \epsilon)$ for all large n. Together with Lemmas 3 and 4, this

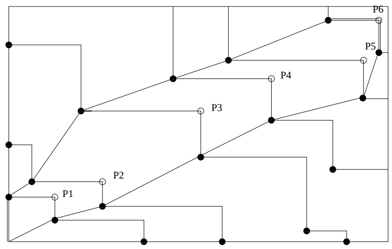

Figure 6. Two geodesics enclosing a beta-path.

yields δ-straightness of β-paths: for all $\delta \in (0, 1/3)$ there exist a constant $c > 0$ so that almost surely

$$ang(P_n, P_m) \leq c|P_n|^{-\delta}$$

for all large n and $m \geq n$, which yields Theorem 3 if $\lambda\rho < 1$. The convergence when $\lambda\rho = 1$ follows from Remark 3. □

Proof of Theorems 1 and 2. The almost sure convergence of X_t/t as well as φ_s/s follows from Theorem 3, since we have already noticed that the trajectories of the second class particles and of the competition interface are β-paths. The limit distribution of the second-class particle follows from (2.4). □

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