Longest increasing subsequences

π_n: permutation of 1,2,. . . ,n.

$$\begin{aligned} L_n &= \ell_n(\pi_n) \\ &= \text{length of a \textit{longest increasing}} \\ &\quad \text{\textit{subsequence} of } \pi_n. \end{aligned}$$

Example:

$$\pi_n = (\pi_n(1), \ldots, \pi_n(n))$$
$$= (7, 2, 8, 1, 3, 4, 10, 6, 9, 5),\ n = 10.$$

A longest increasing subsequence is:

$$(1, 3, 4, 6, 9).$$

Another longest increasing subsequence is:

$$(2, 3, 4, 6, 9).$$

$$L_n = \ell_n(\pi_n) = 5.$$

Patience sorting

Take a deck of cards labeled $1, 2, \ldots, n$. After shuffling, cards are turned up one at a time and dealt into piles on the table, using the rule:

Sorting rule: A low card may be placed on a higher card, or may be put into a new pile to the right of the existing piles.

Greedy strategy: Always place a card on the leftmost possible pile.

For the permutation $(7, 2, 8, 1, 3, 4, 10, 6, 9, 5)$ the greedy strategy yields the stacks:

1			5	
2	3		6	
7	8	4	10	9

Lemma 1 *$L_n = \#$ stacks at end of game.*

Proof: If $a_1 < a_2 < \ldots < a_k$, then, according to the rules of the game, a_{i+1} has to be in some pile to the right of the pile containing a_i, since cards on top of a_i have to be smaller than a_i itself.
So: number of stacks $\geq L_n$.
Conversely, using the greedy strategy, if a card is placed in a pile other than the first pile, put a pointer from that card to the current top card of the preceding pile. At the end of the game, let a_k be the card on top of the rightmost pile. The sequence $a_1 \leftarrow a_2 \leftarrow \ldots \leftarrow a_k$ obtained by following the pointers, is an increasing subsequence with length equal to the number of piles.

□

Note: The top cards do not necessarily give a longest increasing subsequence, since they do not have to correspond to an increasing subsequence of the original permutation!

An interacting particle process

Start with zero particles. At each step, pick a random point U in $[0, 1]$; simultaneously, let the nearest particle (if any) to the right of U disappear.
Lemma 1 implies: the number of particles after n steps is distributed as L_n (Hammersley (1972)).

Continuous time version: let new particles appear according to a Poisson process of rate 1.

Evolution rule: (Aldous and Diaconis (1996): At times of a Poisson (rate x) process in time, a point U is chosen uniformly on $[0, x]$, independent of the past, and the particle nearest to the right of U is moved to U, with a new particle created at U if no such particle exists in $[0, x]$.

Alternative descriptions

Start with a Poisson point process of intensity 1 on $\mathbb{R}^2_+$. Now shift the positive x-axis vertically through (a realization of) this point process, and, each time a point is caught, shift to this point the previously caught point that is immediately to the right.

Let $N(x, y)$ be the number of particles at time y in the interval $[0, x]$. Then (Poissonization!):

$$L_{\widetilde{N}_{x,y}} = N(x, y),$$

where

$$\begin{aligned} \widetilde{N}_{x,y} &= \#\{\text{points of Poisson} \\ &\qquad\qquad \text{point process in } [0,x] \times [0,y]\} \\ &\sim \ \text{Poisson}(xy). \end{aligned}$$

Another equivalent description:

$N(x, y)$ is the maximal number of points on a North-East path from $(0, 0)$ to (x, y) with vertices at the points of the Poisson point process.

Aldous and Diaconis (1995) call the process $y \mapsto N(\cdot, y)$, $y \geq 0$, *Hammersley's interacting particle process*.

Some history

Erdös and Szekeres (1935): $EL_n \geq \frac{1}{2}\sqrt{n}$
Ulam (1961): Monte-Carlo simulation for $n = 1, \ldots, 10$:

$$EL_n \sim 1.7\sqrt{n}.$$

Ulam conjectured that

$$\lim_{n\to\infty} EL_n/\sqrt{n} = c \tag{1}$$

exists.

Ulam's problem: prove that the limit in (1) exists and compute c.
It was proved in Hammersley (1972) that, as $n \to \infty$,

$$L_n/\sqrt{n} \xrightarrow{p} c,$$

where $\xrightarrow{p}$ denotes convergence in probability, and

$$\lim_{n\to\infty} EL_n/\sqrt{n} = c,$$

Hammersley also proved:

$$\pi/2 \le c \le e.$$

Basic tool in Hammersley (1972): subadditive ergodic theorem, using

$$N\left([r,t]^2\right) \ge N\left([r,s]^2\right) + N\left([s,t]^2\right), 0 \le r < s < t.$$

See, e.g., Durrett (1991).

Subsequently Kingman (1973) showed that

$$1.59 < c < 2.49,$$

and later work by Logan and Shepp (1977) and Vershik and Kerov (1977, 1981, 1985) showed that actually $c = 2$.

Aldous and Diaconis (1995) use a "soft hydrodynamical argument", based on properties of Hammersley's interacting particle process.

Central result in Aldous and Diaconis (1995):

Theorem 1 *Let N be Hammersley's process on $\mathbb{R}_+$, started from the empty configuration on the axes. Then, For each fixed $a > 0$, the random particle configuration with counting process*

$$y \mapsto N(t + y, at) - N(t, at) : y \geq -t,$$

converges in distribution, as $t \to \infty$, to a homogeneous Poisson process on $\mathbb{R}$, with intensity $\sqrt{a}$.

Proof uses the ergodic decomposition theorem and and a construction of Hammersley's process on the whole real line. It is first proved that the only invariant point processes are mixed Poisson and next that the random intensity is in fact a non-random intensity if one goes to ∞ along a "ray" $y = ax$.

Stationary version of Hammersley's process

Poisson "sources" on the positive x-axis, Poisson "sinks" on the positive y-axis. If intensity of sources x-axis is a, intensity of sinks in $1/a$, intensity of Poisson process in $\mathbb{R}_+^2$ (all independent), then the expected length of a longest path from $(0,0)$ to (t,t) is $ta + t/a$.

$\Rightarrow$ If $a = 1$, expected length is $2t$.

Rains, Spohn,..

Proof uses Lemma 2.

Lemma 2 *(Groeneboom (2002)).*
Let for a continuous function with compact support $f : \mathbb{R}_+ \to \mathbb{R}_+$*, the function* $L_f : \mathcal{N} \to \mathbb{R}_+$ *be defined by*

$$L_f(\xi) = \exp\{-\xi(f)\}.$$

If ξ_0 *is a Poisson point process of sources with intensity* a *and the sinks are a Poisson point process of sources with intensity* $1/a$*, while the Poisson point process in the interior of* $\mathbb{R}_+^2$ *have intensity 1 (all three processes independent),*

$$E[\mathcal{G}L_f](\xi_0) = 0,$$

where $\mathcal{G}$ *is the infinitesimal generator of the process (of point processes on* $[0,1]$*).*

$\Rightarrow$ The process is stationary in time
$\Rightarrow$ The crossings of horizontal lines are Poisson with intensity a.
$\Rightarrow$ Expected length of longest North-East path from $(0,0)$ to (t,t) is expected number of "sinks" on segment $\{0\} \times [0,t]$ plus expected number of crossings of segment $[0,t] \times \{t\}$ is $t/a + at$.

Key idea: If we start with the stationary Hammersley process and omit the sinks one by one, there is a "separating shock wave", separating stationary and non-stationary behavior of the process.

Totally asymmetric simple exclusion processes (TASEP).

Start with a configuration of particles on $\mathbb{Z}$. Each particle waits (independently of the other particles) an exponentially distributed time and then attempts to jump to the site immediately to its right. If the site is empty, it jumps to this site, otherwise it does not move.

Invariant measures: convex combinations of translation invariant measures and "blocking measures".
Translation invariant measures: measures ν_ρ, putting at each site independently a particle with probability ρ and no particle with probability $1-\rho$.
Blocking measures: concentrated on configurations with particles to the right of a given site and no particles to its left.

Shock waves: If we start with a lower density to the left of zero, we get a shock wave, originating from zero.

Second class particle identifies the shock.
Start with the measure $\nu_{\rho,\lambda}$ on $\mathbb{Z} \setminus \{0\}$, $\rho < \lambda$: density ρ left of zero, density λ right of zero. Put a "2nd class particle" $X(0)$ at 0.
Call the 2nd class particle $X(t)$ at time $t > 0$.
$X(t)$ may jump to the right if that site is empty. Other (first class) particles treat the location of $X(t)$ as an empty spot.
If a first class particle jumps to the site of $X(t)$, $X(t)$ jumps to the (leaving) site of this particle.
Ferrari et al.:

Theorem 2

$$t^{-1}X(t) \xrightarrow{a.s.} 1 - \rho - \lambda,\ t \to \infty.$$

Moreover, if μ_t is the distribution of the particle process, as seen from $X(t)$ at time t, then, uniformly in t:

$$\lim_{x\to\infty} \tau_x\mu_t = \nu_\lambda,\ \lim_{x\to-\infty} \tau_x\mu_t = \nu_\rho,$$

(in the sense of weak convergence), where

$$\tau_x\mu_t(\cdot) = \mu_t(x + \cdot).$$

Starting measure $\nu_{\alpha,\alpha}$ on $\mathbb{Z}\setminus\{0\}$:
density α left of zero, density α right of zero.
Put a 2nd class particle $X(0)$ at 0.
Call the 2nd class particle $X(t)$ at time $t>0$.
Let η_t denotes process at time t, and

$$u(x,t) = P\left\{\eta_t(\{x\}) = 1\right\}.$$

$u(x,t)$ satisfies approximately the equation

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= u(x-1,t)\{1-u(x,t)\} \\ &\qquad -u(x,t)\{1-u(x+1,t)\},\end{aligned}$$

This is a discrete approximation to the Burgers' equation

$$u_t(x,t) + \{1-2u(x,t)\}u_x(x,t) = 0. \tag{2}$$

Corollary 1 *Let $X(t)$ be the location of the 2nd class particle at time t. Then:*

$$t^{-1}X(t) \xrightarrow{\text{a.s.}} 1-2\alpha,\ t\to\infty,$$

that is: $X(t)$ travels asymptotically along a characteristic of the Burgers' equation (2).

Back to Hammersley's process.
Consider the original Hammersley process (without sources or sinks).
$w(x,t) = EN(x,t)$: the expected number of crossings of $[0,x]\times\{t\}$.
The Aldous-Diaconis result means that, along a "ray" $t = ax$, $w(x,t)$ satisfies asymptotically the equation

$$\frac{\partial w(x,t)}{\partial t} = 1/\sqrt{a} = 1\Big/\frac{\partial w(x,t)}{\partial x}\,. \tag{3}$$

This means that the *density* $u(x,t) = \frac{\partial w(x,t)}{\partial x}$ satisfies asymptotically the equation

$$u_t(x,t) + u(x,t)^{-2}u_x(x,t) = 0. \tag{4}$$

So $u(x,t)$ converges asymptotically to the constant $\sqrt{a}$ along $\{(x,t) = (t/a,t) : t \ge 0\}$ (i.e., a *characteristic* of the equation (4)).
In the stationary process with sources of intensity $\sqrt{a}$ and sinks of intensity $1/\sqrt{a}$, this is also the asymptotic direction of an isolated 2nd class particle, located at 0 at time 0.
Analogous situation to that of Corollary 1!

Theorem 3 *(CG (2003)) Let N be the stationary Hammersley process, with intensities $\sqrt{a}$ and $1/\sqrt{a}$ on the x- and y-axis, respectively. Let X_t be the position of an isolated second class particle at time t, located at the origin at time zero. Then*

$$t^{-1}X_t \xrightarrow{\text{a.s.}} 1/a,\, t \to \infty. \tag{5}$$

Corollary 2 *Let N be Hammersley's process on $\mathbb{R}_+$, started from the empty configuration on the axes. Then,*
(i) For each fixed $a > 0$, the random particle configuration with counting process

$$y \mapsto N(t+y, at) - N(t, at) : y \geq -t,$$

converges in distribution, as $t \to \infty$, to a homogeneous Poisson process on $\mathbb{R}$, with intensity $\sqrt{a}$.
(ii)

$$\lim_{t\to\infty} EN(t,t)/t = 2.$$

Proof of (i): Couple the original Hammersley process N with Hammersley's process $N_{a'}$ with sources of intensity $\sqrt{a'}, a' > a$, and no sinks. The set of crossings of each horizontal line of the original Hammersley process is contained in the set of crossings for the process with sinks. In the process with sinks there is a "separating wave" of asymptotic slope a', separating stationary (below the wave) from non-stationary (above the wave) behavior.

Hence, for a finite collection of disjoint intervals $[a_i, b_i)$, $i = 1, \ldots, k$, and non-negative numbers $\lambda_1, \ldots, \lambda_k$, we obtain:

$$\begin{aligned}&Ee^{-\sum_{i=1}^k \lambda_i\{N(t+b_i,at)-N(t+a_i,at)\}}\\ &\geq Ee^{-\sum_{i=1}^k \lambda_i\left\{N_{a'}^{-y}(t+b_i,at)-N_{a'}^{-y}(t+a_i,at)\right\}}.\end{aligned}$$

But the right side converges by Theorem 3 to

$$e^{-\sum_{i=1}^k (b_i-a_i)\sqrt{a'}\left\{1-e^{-\lambda_i}\right\}}.$$

Upper bound follows by coupling with a process with sinks and no sources.

Proof of (ii): The original Hammersley process has to satisfy

$$EN(t,t) \le 2t,$$

since the length of a longest path from $(0,0)$ to (t,t) in this process is always smaller than or equal to the length of a (weakly) longest path from $(0,0)$ to (t,t) in the stationary version of the process, with intensity 1 on the axes.
For the other (asymptotic) inequality, we take a partition $0, t/k, 2t/k, \dots, t$ of the interval $[0,t]$. The crossings of the segment $[(i-1)t/k, it/k] \times \{t\}$ contain the crossings of this line segment by the paths of a Hammersley process N_{a_i} with sinks of intensity $1/\sqrt{a_i}, a_i < k/i$, on the y-axis, but no sources on the x-axis.
Hence, for all comparison processes with sinks of intensity $1/\sqrt{a_i}$, $a_i < k/i$, for the ith interval of the partition, we find:

$$\liminf_{t\to\infty} EN(t,t)/t \ge \frac{1}{k}\sum_{i=1}^{k}\sqrt{a_i}.$$

Thus

$$\liminf_{t\to\infty} EN(t,t)/t \geq \sum_{i=1}^{k} 1/\sqrt{ik} = 2(1+O(1/k)),$$

and (ii) follows.

□

Neither the subadditive ergodic theorem nor the ergodic decomposition theorem is needed!

The original proof in CG (2003)) used that, in the stationary Hammersley process, the "beta points" (North-East corners of the space-time paths) are Poisson.
This is a kind of 2-dimensional Burke theorem.
The 2-dimensional Burke theorem was used to apply time-reversal.

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