Cube-root asymptotics for Hammersley's process

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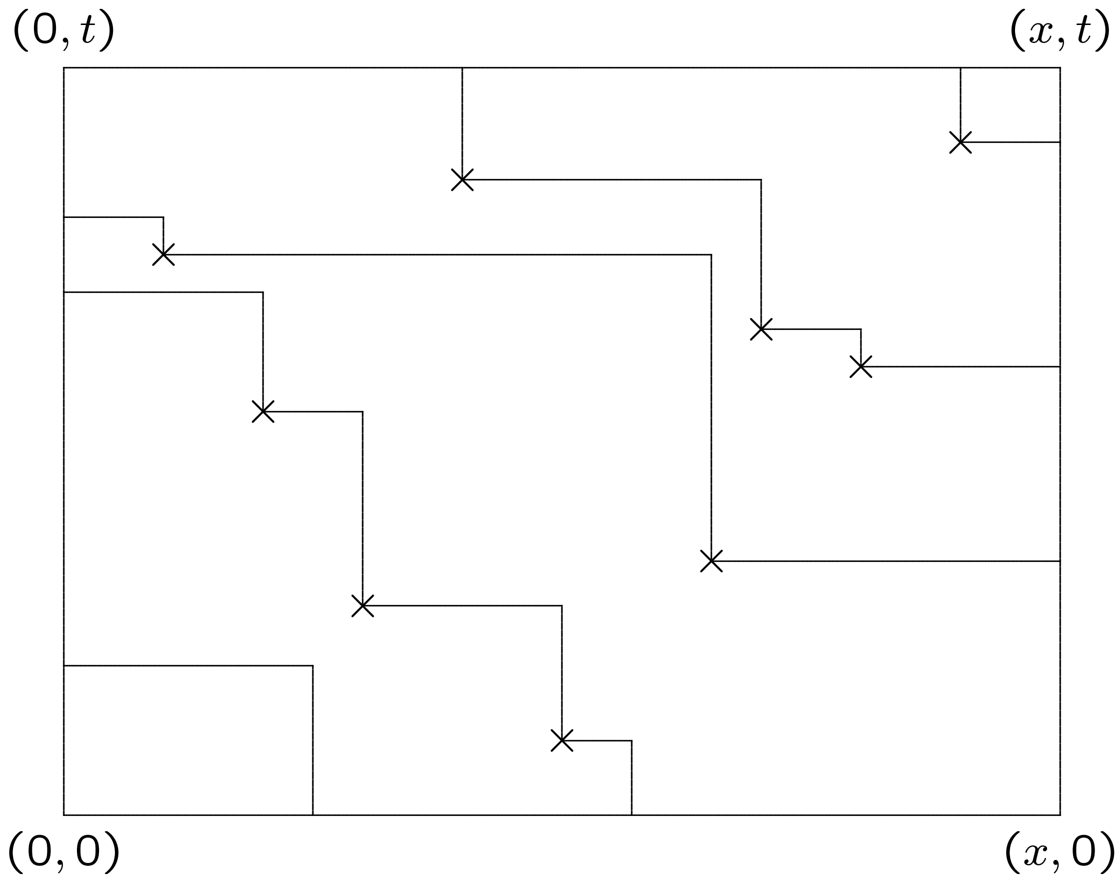
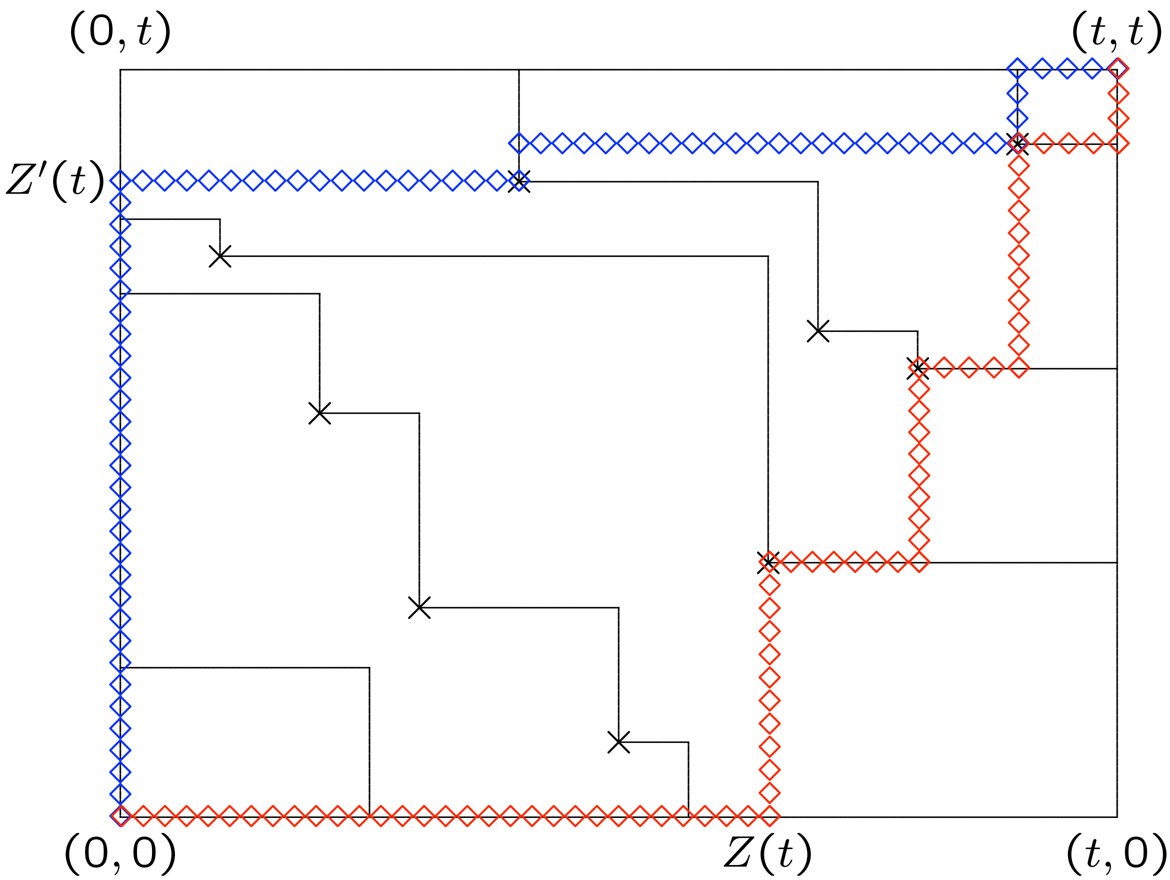
Hammersley's process with sources and sinks:



Sources are Poisson with intensity λ.
Sinks are Poisson with intensity $1/\lambda$.
α-points ($\times$) are Poisson with intensity 1.

Consider symmetric case ($\lambda = 1$).

Longest weakly NE paths to (t, t)



Note that here $Z'(t) < 0$. Furthermore, $Z(t) \stackrel{\mathcal{D}}{=} -Z'(t)$.

Define $L(x,t)$ as the length of a longest weakly NE path to (x,t) in the symmetric case.

Theorem:

$$\text{Var}(L(x,t)) = -x + t + 2\mathbb{E}Z_+.$$

Proof: Introduce 4 directions S, W, N and E.

$$L(x,t) = S + E = N + W.$$

Burke's Theorem states that N and E are independent, so

$$\begin{aligned} \text{Var}(L(x,t)) &= \text{Var}(W) - \text{Var}(N) + 2\text{Cov}(N,S) \\ &= -x + t + 2\text{Cov}(N,S). \end{aligned}$$

Take a source-intensity $1+\varepsilon$. Condition on S:

$$\text{Cov}(N,S) = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \mathbb{E}_\varepsilon(N) = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \mathbb{E}_\varepsilon(N+W).$$

This derivative corresponds to adding an extra source uniformly on $[0,x]$. An extra source at y only leads to an increase of $L(x,t)$ if $y \leq Z_+$.

Rescaling and monotonicity gives for $\lambda \geq 1$

Corollary

$$\mathrm{Var}(L_\lambda(t,t)) \leq (\lambda - 1/\lambda)t + \mathrm{Var}(L(t,t)).$$

Important because of following idea. Define

$$N(z) = \text{number of sources in } [0,z] \times \{0\}$$

and

$$A_t(z) = \text{length of longest strictly NE path from } (z,0) \text{ to } (t,t).$$

Then N and A_t are independent processes and

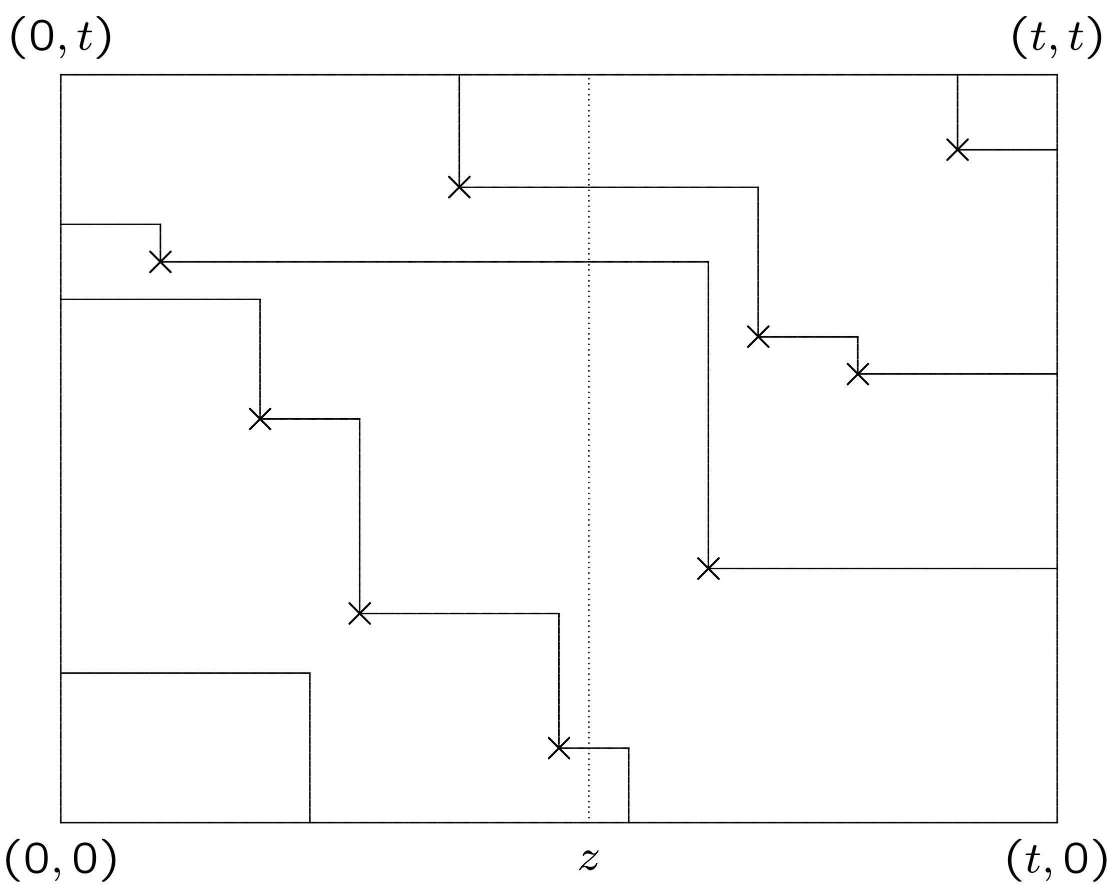
$$L(t,t) = \sup\{N(z) + A_t(z) : -t \leq z \leq t\}.$$

Note that N is just a Poisson process. Furthermore, we can control the process A_t.

L_λ has thickened set of sources ($\lambda > 1$) and thinned set of sinks ($1/\lambda$).

Lemma:

$$A_t(z) \leq L_\lambda(t,t) - L_\lambda(z,0).$$



$$\mathbb{E}A_t(z) \leq 2\sqrt{t(t-z)} \approx 2t - z - z^2/4t.$$

$$
\begin{aligned}
&\mathbb{P}\ \{Z(t) > u\} \\
&\quad = \mathbb{P}\{\exists\ z > u : N(z) + A_t(z) = L(t,t)\} \\
&\quad \le \mathbb{P}\{\exists\ z > u : N(z) + L_\lambda(t,t) - L_\lambda(z,0) \ge L(t,t)\} \\
&\quad = \mathbb{P}\{\exists\ z > u : L_\lambda(z,0) - N(z) \le L_\lambda(t,t) - L(t,t)\}.
\end{aligned}
$$

Note that $\tilde{N}(z) = L_\lambda(z,0) - N(z)$ is a Poisson process with intensity $\lambda - 1$. Therefore

$$\mathbb{P}\{Z(t) > u\} \le \mathbb{P}\{\tilde{N}(u) \le L_\lambda(t,t) - L(t,t)\}.$$

Optimize λ:

$$\lambda = (1 - u/t)^{-1/2}.$$

$$\mathbb{E}\tilde{N}(u) - \mathbb{E}\{L_\lambda(t,t) - L(t,t)\} \ge \frac{1}{4}u^2/t.$$

$$\mathrm{Var}\tilde{N}(u) \le u.$$

$$\mathrm{Var}(L_\lambda(t,t) - L(t,t)) \le 8\mathbb{E}Z(t)_+ + 4u.$$

Chebyshev:

$$\mathbb{P}\{Z(t) > u\} \lesssim \frac{t^2}{u^3} + \frac{t^2 \mathbb{E}Z(t)_+}{u^4}.$$

Take $u = c\,\mathbb{E}Z(t)_+$. Then

$$\mathbb{P}\{Z(t) > c\,\mathbb{E}Z(t)_+\} \lesssim \frac{t^2}{\mathbb{E}Z(t)_+^3}\left(\frac{1}{c^3} + \frac{1}{c^4}\right).$$

If

$$\frac{\mathbb{E}Z(t_n)_+^3}{t_n^2} \to \infty,$$

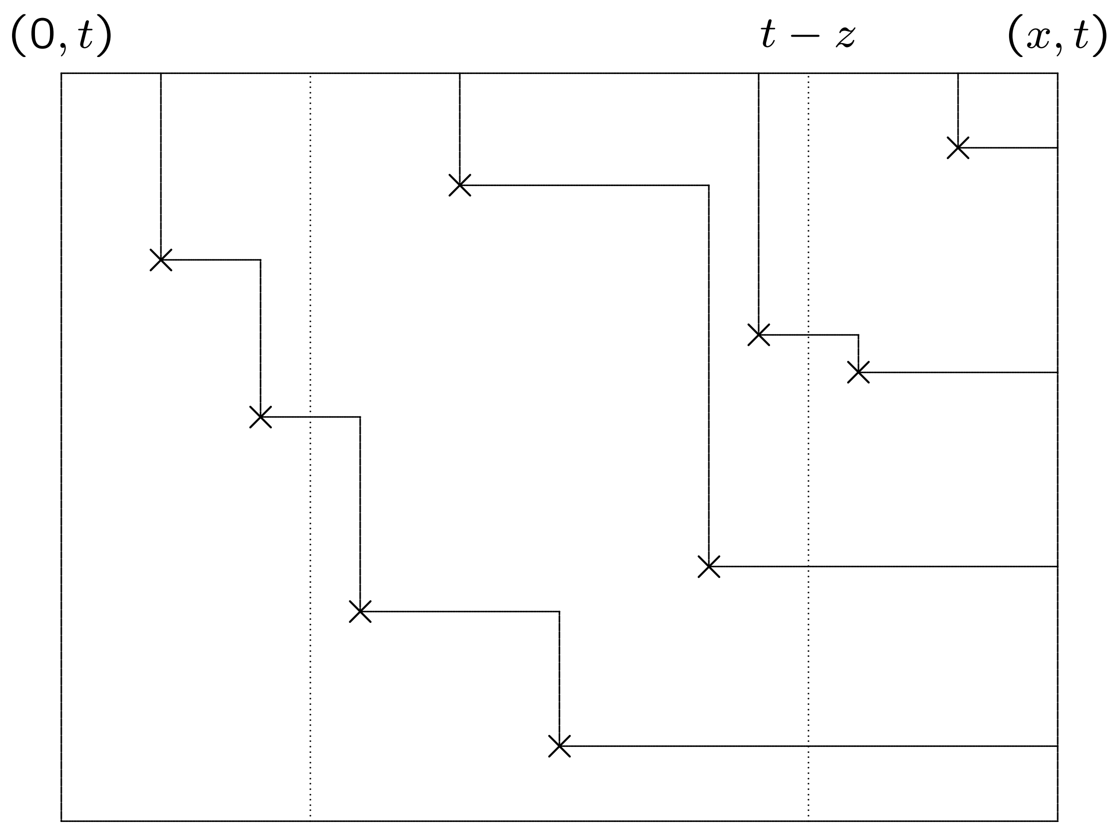
then Dominated Convergence shows that

$$\mathbb{E}\left(\frac{Z(t_n)_+}{\mathbb{E}Z(t_n)_+}\right) \to 0,$$

which is absurd. Therefore,

$$\limsup_{t\to\infty} \frac{\mathbb{E}Z(t)_+}{t^{2/3}} < +\infty.$$

Consider the original Hammersley process. $L_0(x,t)$ is length of longest path to (x,t).



$$A_t(0) - A_t(z) \stackrel{\mathcal{D}}{=} L_0(t,t) - L_0(t-z,t).$$

Corollary:

$$\liminf_{t \to \infty} \frac{\mathbb{E}Z(t)_+}{t^{2/3}} > 0.$$

Suppose $t_n \to \infty$ such that

$$\frac{\mathbb{E}Z(t_n)_+}{t_n^{2/3}} \to 0.$$

Then

$$\mathbb{P}\{Z(t_n) > \varepsilon t_n^{2/3}\} \;\; \leq \;\; \frac{\mathbb{E}Z(t_n)_+}{\varepsilon t_n^{2/3}} \to 0.$$

Since $-Z'(t) \stackrel{\mathcal{D}}{=} Z(t)$ and $Z'(t) \leq Z(t)$, we have that $\mathbb{P}\{Z(t) \geq 0\} \geq 1/2$. This would mean that for all $\varepsilon > 0$,

$$\liminf_{n \to \infty} \mathbb{P}\{0 \leq Z(t_n) \leq \varepsilon t_n^{2/3}\} \geq \frac{1}{2},$$

which contradicts the previous Theorem.