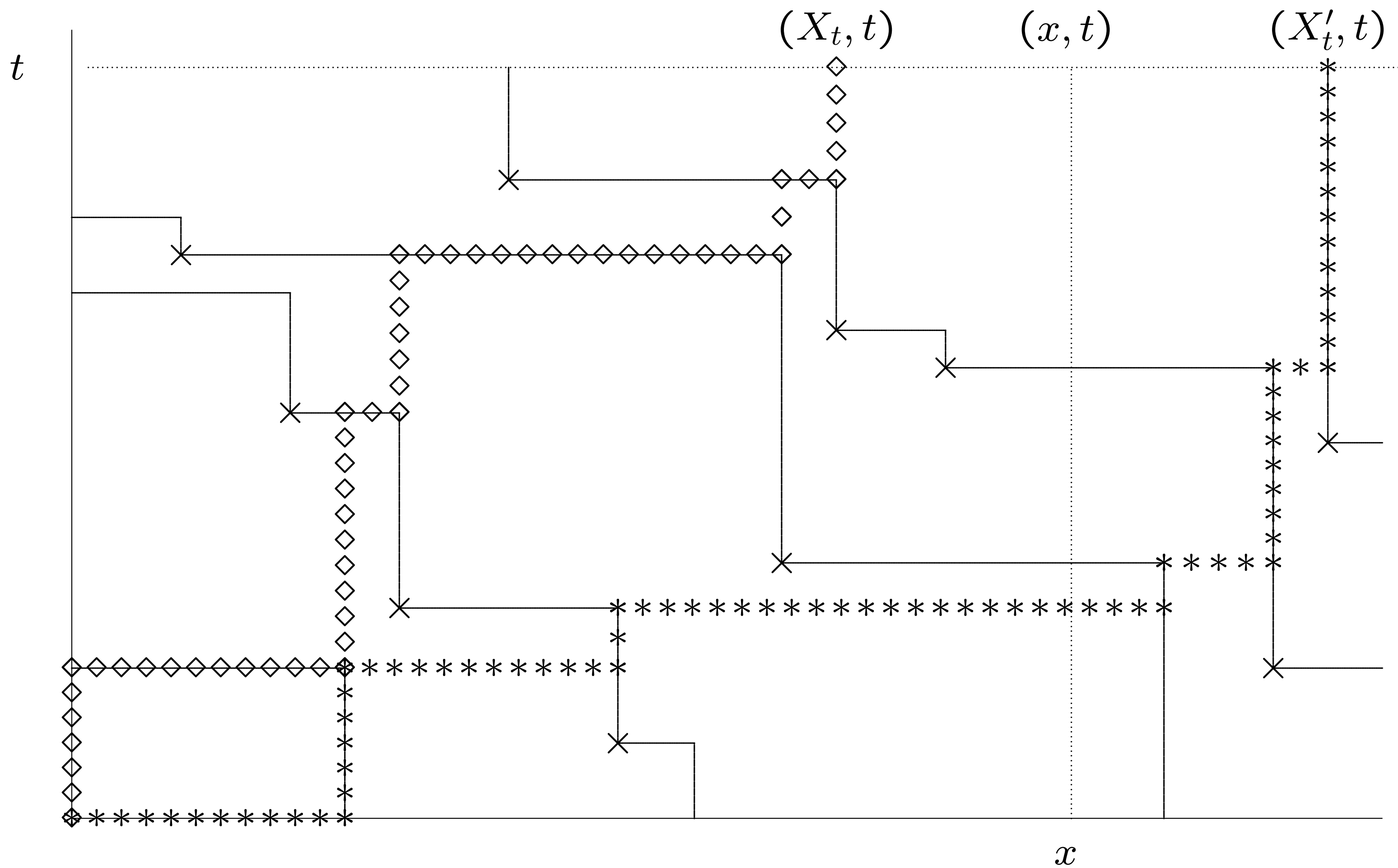
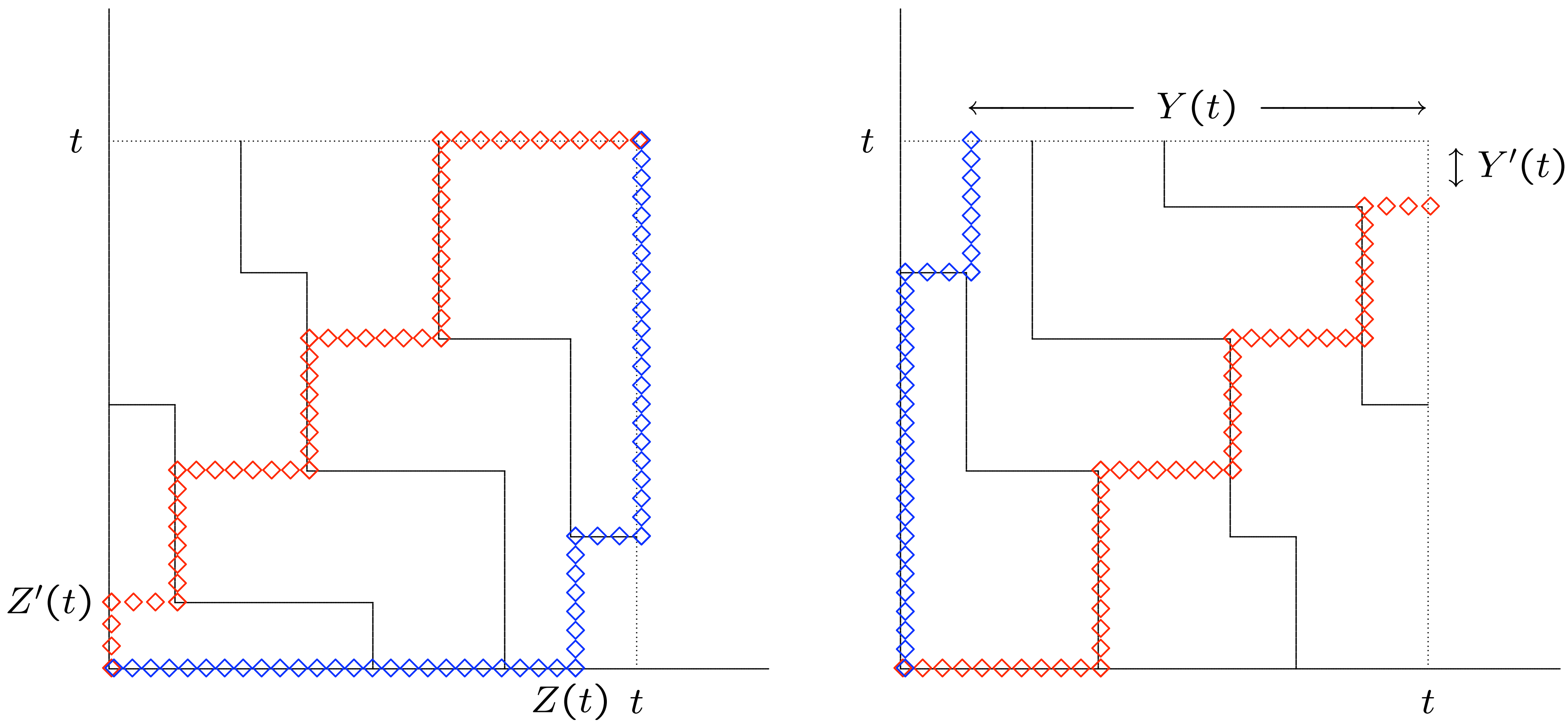
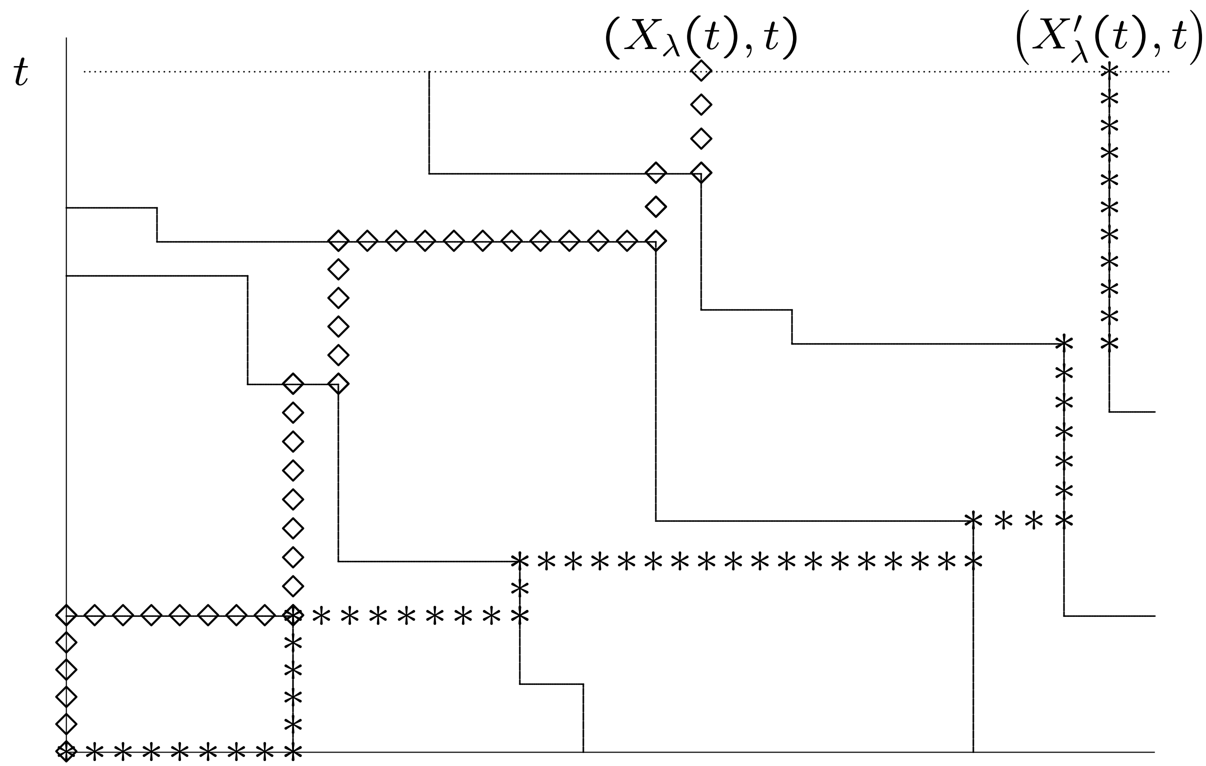

t
(X_t, t)
(x, t)
(X'_t, t)
x



$$(Y'(t), Y(t)) \stackrel{\mathcal{D}}{=} (Z'(t), Z(t)).$$



$$x \le y \le X'_\lambda(t) \Rightarrow L_\lambda(y,t) - L_\lambda(x,t) \le L_0(y,t) - L_0(x,t).$$

$$y \ge x > X_\lambda(t) \;\; \Rightarrow L_0(y,t) - L_0(x,t) \le L_\lambda(y,t) - L_\lambda(x,t).$$

Theorem:

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty}\mathbb{P}\left\{0 \le Z(t) \le \varepsilon t^{2/3}\right\} = 0.$$

Proof:

$$\mathbb{P}\left\{0 \le Z(t) \le \varepsilon t^{2/3}\right\}$$

$$\le \mathbb{P}\left\{\sup_{z>\varepsilon t^{2/3}}(N(z)+A_t(z)) < \sup_{z\in[0,\varepsilon t^{2/3}]}(N(z)+A_t(z))\right\}$$

$$\le \mathbb{P}\left\{\sup_{z>\varepsilon t^{2/3}}\{N(z)-(A_t(0)-A_t(z))\} < Lt^{1/3}\right\}$$

$$+\ \mathbb{P}\left\{\sup_{z\in[0,\varepsilon t^{2/3}]}\{N(z)-(A_t(0)-A_t(z))\} > Lt^{1/3}\right\}.$$

Choose $\lambda = 1 - rt^{-1/3}$ and define

$$\tilde{N}_\lambda(z) = L_\lambda(t,t) - L_\lambda(t-z,t).$$

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{z\in[0,\varepsilon t^{2/3}]} \{N(z) - (A_t(0) - A_t(z))\} \geq Lt^{1/3}\right\} \\
&= \mathbb{P}\left\{\sup_{z\in[0,\varepsilon t^{2/3}]} \{N(z) - (L_0(t,t) - L_0(t-z,t))\} \geq Lt^{1/3}\right\} \\
&\leq \mathbb{P}\left\{\sup_{z\in[0,\varepsilon t^{2/3}]} \left\{N(z) - \tilde{N}_\lambda(z)\right\} \geq Lt^{1/3}\right\} + \mathbb{P}\left\{X'_\lambda(t) < t\right\}.
\end{aligned}$$

Second term:

$$\begin{aligned}
\mathbb{P}\left\{X'_\lambda(t) < t\right\} &= \mathbb{P}\left\{X'(t/\lambda) < \lambda t\right\} \\
&= \mathbb{P}\left\{Z'(t/\lambda) > t(1/\lambda - \lambda)\right\} \\
&\le \mathbb{P}\{Z(t/\lambda) > t(1/\lambda - \lambda)\} \\
&= \mathbb{P}\left\{Z\left(t/\left(1 - rt^{-1/3}\right)\right) > rt^{2/3}\left(2 - rt^{-1/3}\right)/\left(1 - rt^{-1/3}\right)\right\} \\
&\le \mathbb{P}\left\{Z\left(\tilde{t}\right) > r\tilde{t}^{\,2/3}\right\} \lesssim r^{-3}.
\end{aligned}$$

First term:

$$\mathbb{P}\left\{\exists_{0\le z\le\varepsilon} : t^{-1/3}\left\{N(zt^{2/3}) - \tilde{N}_\lambda(zt^{2/3})\right\} \ge L\right\}.$$

This process converges to

$$W_r(z) \stackrel{\text{def}}{=} W(2z) + rz,\ z \ge 0.$$

We now get, for $r < L/\varepsilon$,

$$\begin{aligned}
\mathbb{P}\left\{\sup_{z\in[0,\varepsilon]} W_r(z) \geq L\right\} &\leq \mathbb{P}\left\{\sup_{z\in[0,2\varepsilon]} W(z) \geq L - \varepsilon r\right\} \\
&= \mathbb{P}\left\{\sup_{z\in[0,1]} W(2\varepsilon z)/\sqrt{2\varepsilon} \geq (L-\varepsilon r)/\sqrt{2\varepsilon}\right\} \\
&= \mathbb{P}\left\{\sup_{z\in[0,1]} W(z) \geq (L-\varepsilon r)/\sqrt{2\varepsilon}\right\} \\
&= \sqrt{\frac{2}{\pi}} \int_{(L-\varepsilon r)/\sqrt{2\varepsilon}}^{\infty} e^{-\frac{1}{2}u^2}\, du.
\end{aligned}$$

Taking $r = L/(2\varepsilon)$ we get:

$$\sqrt{\frac{2}{\pi}} \int_{(L-\varepsilon r)/\sqrt{2\varepsilon}}^{\infty} e^{-\frac{1}{2}u^2}\, du = 2\int_{L/\sqrt{8\varepsilon}}^{\infty} \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}\, du \sim \frac{2\sqrt{8\varepsilon}\, e^{-L^2/(16\varepsilon)}}{L\sqrt{2\pi}}, \ \varepsilon \downarrow 0.$$