An invariance result for Hammersley's process with sources and sinks

Piet Groeneboom

Delft University of Technology, Vrije Universiteit, Amsterdam, and University of Washington, Seattle

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Abstract

Let $\mathcal{P}_1$ be a Poisson process of intensity λ_1 on the positive x-axis, $\mathcal{P}_2$ a Poisson process of intensity λ_2 on the positive y-axis, and $\mathcal{P}$ a Poisson process of intensity $\lambda_1\lambda_2$ in the interior of $\mathbb{R}_+^2$, where $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}$ are independent. Then the extended Hammersley process $L_{\lambda_1}(\cdot,t)$ with sources and sinks given by $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, is distributed as a Poisson point process with intensity λ_1 for all $t \ge 0$.

1 The invariance result

Let ξ denote a point process on $[0,1]$. That is, ξ is a random (Radon) measure on $[0,1]$, with realizations of the form

$$\xi(f) \stackrel{\text{def}}{=} \int_0^1 f(x)\,d\xi(x) = \sum_{i=1}^N f(\tau_i), \tag{1.1}$$

where $\tau_1,\dots,\tau_N$ are the points of the point process ξ and f is a bounded measurable function $f:(0,1)\to\mathbb{R}_+$. If $N=0$, we define the right side of (1.1) to be zero.

We can consider the random measure ξ as a random sum of Dirac measures:

$$\xi = \delta_{\tau_1} + \dots \delta_{\tau_N}, \tag{1.2}$$

and hence

$$\xi(B) = \sum_{i=1}^N \delta_{\tau_i}(B) = \sum_{i=1}^N 1_B(\tau_i),$$

for Borel sets $B \subset (0,1)$. So $\xi(B)$ is just the number of points of the point process ξ, contained in B, where the sum is defined to be zero if $N=0$. The realizations of a point process, applied on

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Borel subsets of $[0,1]$, take values in $\mathbb{Z}_+$ and belong to a strict subset of the Radon measures on $[0,1]$. We will denote this subset, corresponding to the point processes, by $\mathcal{N}$, and endow it with the vague topology of measures on $[0,1]$, see, e.g., Kallenberg (1986), p. 32. For this topology, $\mathcal{N}$ is a (separable) *Polish space* and a closed subset of the set of Radon measures on $[0,1]$, see Proposition 15.7.7 and Proposition 15.7.4, pp. 169-170, Kallenberg (1986). Note that, by the compactness of the interval $[0,1]$, the *vague topology* coincides with the *weak topology*, since all continuous functions $f:[0,1]\to I\!R$ have compact support, contained in the compact interval $[0,1]$. For this reason we will denote the topology on $\mathcal{N}$ by the *weak* topology instead of the *vague* topology in the sequel. Note that the space $\mathcal{N}$ is in fact *locally compact* for the weak topology.

In our case we have point processes ξ_t, for each time $t\ge 0$, of the form

$$\xi_t=\delta_{\tau_1}+\ldots\delta_{\tau_{N_t}},$$

where N_t denotes the number of points at time t, and where $0\le\tau_1\le\ldots\le\tau_{N_t}<1$; ξ_t is defined to be the zero measure on $[0,1]$, if $N_t=0$. Note that, if we start with a Poisson process of intensity $\lambda_1>0$, the initial configuration ξ_0 will with probability one be either the zero measure or be of the form:

$$\sum_{i=1}^{n}\delta_{\tau_i},\ 0<\tau_1<\ldots<\tau_n<1, \tag{1.3}$$

for some $n>0$, but since we want to consider the space of bounded continuous functions on $\mathcal{N}$, it is advantageous to allow configurations where some τ_i's will be equal. We also allow the τ_i's to take values at 0 or 1. If we have a "stack" of τ_i's at the same location in $(0,1]$, we move one point ("the point on top") from the stack to the left, if a point immediately to the left of the location of the stack appears, leaving the other points at the original location. Likewise, if a stack of τ_i's is located at 0, we remove the point on top of the stack at time t if the Poisson point process on the left lower boundary has a point at $(0,t)$.

Now let $\mathcal{F}_c$ be the Banach space of continuous bounded functions $\phi:\mathcal{N}\to I\!R$ with the supremum norm. For $\phi\in\mathcal{N}$ and $t>0$ we define the function $P_t\phi:\mathcal{N}\to I\!R$ by

$$[P_t\phi](\xi)=E\left\{\phi(\xi_t)\mid\xi_0=\xi\right\}.$$

We want to show that the operator P_t is a mapping from $\mathcal{F}_c$ into itself.

Boundedness of $P_t\phi$ is clear if $\phi:\mathcal{N}\to I\!R$ is bounded and continuous, so we only must prove the continuity of $P_t\phi$, if ϕ is a bounded continuous function $\phi:\mathcal{N}\to I\!R$. If ξ is the zero measure and (ξ_n) is a sequence of measures in $\mathcal{N}$, converging weakly to ξ, we must have:

$$\lim_{n\to\infty}\xi_n([0,1])=0,$$

and hence $\xi_n([0,1])=0$, for all large n. This implies that

$$E\left\{\phi(\xi_t)\mid\xi_0=\xi\right\}=E\left\{\phi(\xi_t)\mid\xi_0=\xi_n\right\},$$

for all large n. If $\xi=\sum_{i=1}^m\delta_{\tau_i}$, with $m>0$, and (ξ_n) is a sequence of measures in $\mathcal{N}$, converging weakly to ξ, we must have $\xi_n([0,1])=m=\xi([0,1])$ for all large n, so ξ_n is of the form

$$\xi_n=\sum_{i=1}^{m}\delta_{\tau_{n,i}},$$

for all large n. Moreover, the ordered $\tau_{n,i}$'s have to converge to the ordered τ_i's in the Euclidean topology. Since the x-coordinates of a realization of the Poisson process of intensity $\lambda_1\lambda_2$ in $(0,1)\times(0,t]$ will with probability one be different from the τ_i's, sample paths of the processes $\{\xi_t : t \ge 0\}$, either starting from ξ or from ξ_n, will develop in the same way, if n is sufficiently large, for such a realization of the Poisson process in $(0,1)\times(0,t]$. Hence

$$\lim_{n\to\infty} E\left\{\phi(\xi_t) \mid \xi_0 = \xi_n\right\} = E\left\{\phi(\xi_t) \mid \xi_0 = \xi\right\}.$$

So we have:

$$\lim_{n\to\infty} [P_t\phi](\xi_n) = [P_t\phi](\xi),$$

if the sequence (ξ_n) converges weakly to ξ, implying the continuity of $P_t\phi$.

Since $\{\xi_t : t \ge 0\}$ is a Markov process with respect to the natural filtration $\{\mathcal{F}_t : t \ge 0\}$, generated by this process, we have the semi-group property

$$P_{s+t}\phi = [P_t \circ P_s]\,\phi,$$

for bounded continuous functions $\phi : \mathcal{N} \to I\!R$. Moreover, we can define the *generator* $\mathcal{G}$ of the process (ξ_t), working on the bounded continuous functions $\phi : \mathcal{N} \to I\!R$ by

$$[\mathcal{G}\phi](\xi) = \lambda_1\lambda_2 \int_0^1 \left\{\phi(\delta_x) - 1\right\}\, dx, \tag{1.4}$$

if ξ is the zero measure on $(0,1)$, and by

$$[\mathcal{G}\phi](\xi) = \lambda_1\lambda_2 \sum_{i=1}^{n+1} \int_{\tau_{i-1}}^{\tau_i} \left\{\phi(\xi^x) - \phi(\xi)\right\}\, dx + \lambda_2 \left\{\phi\left(\sum_{i=2}^{n} \delta_{\tau_i}\right) - \phi(\xi)\right\} \tag{1.5}$$

if $\xi = \sum_{i=1}^n \delta_{\tau_i}$, where $\tau_0 = 0$, $\tau_{n+1} = 1$, and where ξ^x is defined by

$$\xi^x = \begin{cases} \delta_x + \sum_{i=2}^n \delta_{\tau_i} & ,\ \text{if } 0 < x < \tau_1, \\ \sum_{i=1}^{j-1} \delta_{\tau_i} + \delta_x + \sum_{i=j+1}^n \delta_{\tau_i} & ,\ \text{if } \tau_{j-1} < x < \tau_j,\ 1 < j < n \\ \sum_{i=1}^n \delta_{\tau_i} + \delta_x & ,\ \text{if } \tau_n < x < 1. \end{cases} \tag{1.6}$$

The first term on the right of (1.5) corresponds to the insertion of a new point in one of the intervals (τ_{i-1},τ_i) and the shift of τ_i to this new point if the new point is not in the rightmost interval, and the second term on the right of (1.5) corresponds to an "escape on the left". Note that $\mathcal{G}\phi(\xi)$ is computed by evaluating

$$\lim_{h\downarrow 0}\left[\frac{[P_h\phi](\xi) - \phi(\xi)}{h}\right].$$

The definition of $\mathcal{G}$ can be continuously extended to cover the configurations

$$\sum_{i=1}^n \delta_{\tau_i},\ 0 \le \tau_1 \le \ldots \le \tau_n \le 1, \tag{1.7}$$

working with the extended definition of P_t, described above.

So we have a semigroup of operators P_t, working on the Banach space of bounded continuous functions $\phi : \mathcal{N} \to I\!R$, with generator $\mathcal{G}$. It now follows from Theorem 13.35 in RUDIN (1991) that we have the following lemma.

Lemma 1.1 *Let $\mathcal{N}$ be endowed with the weak topology and let $\phi : \mathcal{N} \to I\!R$ be a bounded continuous function. Then we have, for each $t > 0$,*

$$\frac{d}{dt}[P_t\phi] = [\mathcal{G}P_t\phi] = [P_t\mathcal{G}\phi].$$

Proof: It is clear that the conditions (a) to (c) of definition 13.34 in RUDIN (1991) are satisfied, and the statement then immediately follows. □

We will also need the following lemma (this is the real "heart" of the proof).

Lemma 1.2 *Let for a continuous function $f : [0,1] \to I\!R_+$, the function $L_f : \mathcal{N} \to I\!R_+$ be defined by*

$$L_f(\xi) = \exp\{-\xi(f)\}. \tag{1.8}$$

Then:

$$E[\mathcal{G}L_f](\xi_0) = 0, \text{ for all continuous } f : [0,1] \to I\!R_+. \tag{1.9}$$

Proof. We first consider the value of $\mathcal{G}L_f(\xi_0)$ for the case where ξ_0 is the zero measure, i.e., the interval $[0,1]$ contains no points of the point process ξ_0. By (1.4) we then have:

$$\mathcal{G}L_f(\xi_0) = \lambda_1\lambda_2 \int_0^1 \left\{e^{-f(x)} - 1\right\} dx. \tag{1.10}$$

If $\xi_0 = \delta_x$, for some $x \in (0,1)$, we get

$$\mathcal{G}L_f(\xi_0) = \lambda_2\left\{1 - e^{-f(x)}\right\} + \lambda_1\lambda_2\int_0^x \left\{e^{-f(u)} - e^{-f(x)}\right\} du$$
$$+\lambda_1\lambda_2\int_x^1 \left\{e^{-f(x)-f(u)} - e^{-f(x)}\right\} du.$$

Hence:

$$\begin{aligned}
&E\mathcal{G}L_f(\xi_0)1_{\{\xi_0([0,1])\le 1\}}\\
&= \lambda_1\lambda_2 e^{-\lambda_1}\int_0^1 \left\{e^{-f(x)} - 1\right\} dx + \lambda_1\lambda_2 e^{-\lambda_1}\int_0^1 \left\{1 - e^{-f(x)}\right\} dx\\
&\quad -\lambda_1^2\lambda_2 e^{-\lambda_1}\int_0^1 ue^{-f(u)}\, du + \lambda_1^2\lambda_2 e^{-\lambda_1}\iint_{0<x<u<1} e^{-f(x)-f(u)}\, dx\, du\\
&= -\lambda_1^2\lambda_2 e^{-\lambda_1}\int_0^1 ue^{-f(u)}\, du + \lambda_1^2\lambda_2 e^{-\lambda_1}\iint_{0<x<u<1} e^{-f(x)-f(u)}\, dx\, du.
\end{aligned}$$

Now generally suppose that, for $n > 1$,

$$E\mathcal{G}L_f(\xi_0)1_{\{\xi_0([0,1])\le n-1\}} \tag{1.11}$$

$$= -\lambda_1^n \lambda_2\, e^{-\lambda_1} \int_{0<x_1<\ldots<x_{n-1}<1} x_1 \exp\left\{-\sum_{i=1}^{n-1} f(x_i)\right\} dx_1 \ldots dx_{n-1}$$

$$+\lambda_1^n \lambda_2\, e^{-\lambda_1} \int_{0<x_1<\ldots<x_n<1} \exp\left\{-\sum_{i=1}^{n} f(x_i)\right\} dx_1 \ldots dx_n. \tag{1.12}$$

Then, by a completely similar computation, it follows that

$$\begin{aligned} &E\mathcal{G}L_f(\xi_0)1_{\{\xi_0([0,1])\le n\}} = E\mathcal{G}L_f(\xi_0)1_{\{\xi_0([0,1])\le n-1\}} + E\mathcal{G}L_f(\xi_0)1_{\{\xi_0([0,1])= n\}} \\ &= -\lambda_1^{n+1} \lambda_2\, e^{-\lambda_1} \int_{0<x_1<\ldots<x_n<1} x_1 \exp\left\{-\sum_{i=1}^{n} f(x_i)\right\} dx_1 \ldots dx_n \\ &\qquad +\lambda_1^{n+1} \lambda_2\, e^{-\lambda_1} \int_{0<x_1<\ldots<x_{n+1}<1} \exp\left\{-\sum_{i=1}^{n+1} f(x_i)\right\} dx_1 \ldots dx_{n+1}. \end{aligned}$$

So we get

$$E\mathcal{G}L_f(\xi_0) = \lim_{n\to\infty} E\mathcal{G}L_f(\xi_0)1_{\{\xi_0([0,1])\le n\}} = 0,$$

since

$$\int_{0<x_1<\ldots<x_n<1} x_1 \exp\left\{-\sum_{i=1}^{n} f(x_i)\right\} dx_1 \ldots dx_n < \frac{1}{n!},$$

and similarly

$$\int_{0<x_1<\ldots<x_{n+1}<1} \exp\left\{-\sum_{i=1}^{n+1} f(x_i)\right\} dx_1 \ldots dx_{n+1} \le \frac{1}{(n+1)!}.$$

□

We now have the following corollary.

Corollary 1.1 *Let $\phi : \mathcal{N} \to I\!R$ be a continuous function with compact support in $\mathcal{N}$. Then:*

$$E[\mathcal{G}\phi](\xi_0) = 0. \tag{1.13}$$

Proof. Let C be the compact support of ϕ in $\mathcal{N}$. The functions L_f, where f is a continuous function $f : [0,1] \to I\!R_+$, are closed under multiplication and hence linear combinations of these functions, restricted to C, form an *algebra.* Since the constant functions also belong to this algebra and the functions L_f separate points of C, the Stone-Weierstrass theorem implies that ϕ can be uniformly approximated by functions from this algebra, see, e.g., Dieudonné (1969), (7.3.1), p. 137. The result now follows from Lemma 1.2, since $\mathcal{G}$ is clearly a bounded continuous operator on the Banach space of continuous functions $\psi : C \to \mathbb{R}$. $\square$

Now let $\phi : \mathcal{N} \to \mathbb{R}$ be a continuous function with compact support in $\mathcal{N}$. Then $P_t \circ \phi$ is also a continuous function with compact support in $\mathcal{N}$, for each $t > 0$. By Corollary 1.1 we have:

$$E[\mathcal{G}P_t\phi](\xi_0) = 0.$$

Hence, by Lemma 1.1,

$$E[P_t\phi](\xi_0) - E\phi(\xi_0) = \int_0^t E[\mathcal{G}P_s\phi](\xi_0)\,ds = 0,\; t > 0,$$

implying

$$E\phi(\xi_t) = E[P_t\phi](\xi_0) = E\phi(\xi_0),$$

for each continuous function $\phi : \mathcal{N} \to \mathbb{R}$ with compact support in $\mathcal{N}$. But since $\mathcal{N}$ is a Polish space, every probability measure on $\mathcal{N}$ is "tight", and hence ξ_t has the same distribution as ξ_0 for every $t > 0$ (here we could also use the fact that $\mathcal{N}$ is in fact locally compact for the weak topology).

Remark. For a general result on stationarity of interacting particle processes (but with another state space!), using an equation of type (1.13), see, e.g., LIGGETT (1985), Proposition 6.10, p. 52.

Corollary 1.2 *Let $\mathcal{P}_1$ be a Poisson process of intensity λ_1 on the positive x-axis, $\mathcal{P}_2$ a Poisson process of intensity λ_2 on the positive y-axis, and $\mathcal{P}$ a Poisson process of intensity $\lambda_1\lambda_2$ in the interior of $\mathbb{R}_+^2$, where $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}$ are independent. Then $L_{\lambda_1}(\cdot, t)$ with sources and sinks given by $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, is distributed as a Poisson point process with intensity λ_1 for all $t \geq 0$.*

It is also clear that the inequality $c \leq 2$ follows, since the length of a longest North-East path from $(0,0)$ to a point (t,t), with $t > 0$, will, in the construction above, be always at least as big as the length of a longest North-East path from $(0,0)$ to a point (t,t), if we start with the empty configuration on the x- and y-axis: we simply have more opportunities for forming a North-East path, if we allow them to pick up points from the x- or y-axis. Since, starting with a Poisson process of intensity 1 in the first quadrant, and (independently) Poisson processes of intensity 1 on the x- and y-axis, the expected length of a longest North-East path to (t,t) will be *exactly* equal to $2t$, according to what we proved above, we obtain from this $c \leq 2$.

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