Sources and sinks and $c = 2$

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Local convergence of Hammersley's process

1 Hammersley's process and its extensions

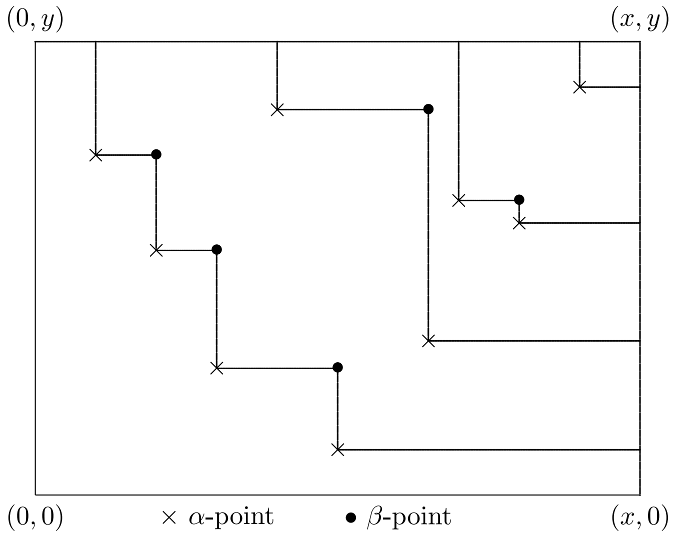


Figure 1: Space-time paths of Hammersley's process, contained in $[0, x] \times [0, y]$.

$L(x, y)$: maximal number of points on a North-East path from $(0, 0)$ to (x, y) with vertices at the points of the Poisson point process in the interior of $\mathbb{R}_+^2$. Picture corresponds to the permutation $(5, 3, 6, 2, 8, 7, 1, 4, 9)$ (ordering via y-coordinates: in paper) or $(7, 4, 2, 8, 1, 3, 6, 5, 9)$ (ordering via x-coordinates). Also: $L(x, y) = \#$ points in $[0, x]$ of Hammersley's process at time y.

We call the points of the Poisson point process in the interior of $\mathbb{R}_+^2$ *α-points* and the North-East corners of the space-time paths of the Hammersley's process *β-points*.

Extension of Hammersley's interacting particle process:
In addition to the Poisson point process in the interior of $\mathbb{R}_+^2$, we have mutually independent Poisson point processes on the x- and y-axis, independent of the Poisson point process in the interior of $\mathbb{R}_+^2$.
Sources: the Poisson point process on the x-axis
Sinks: the Poisson point process on the y-axis

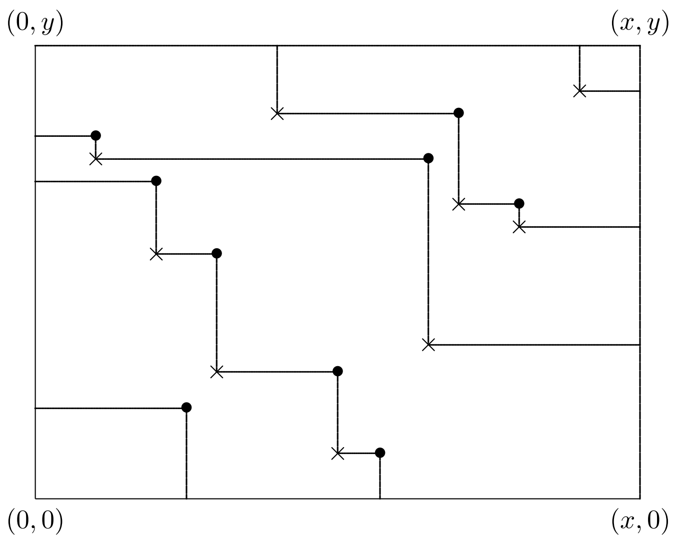
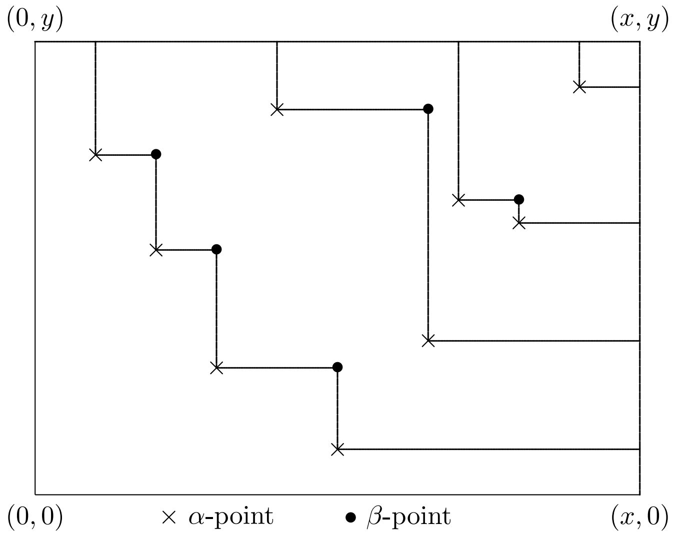
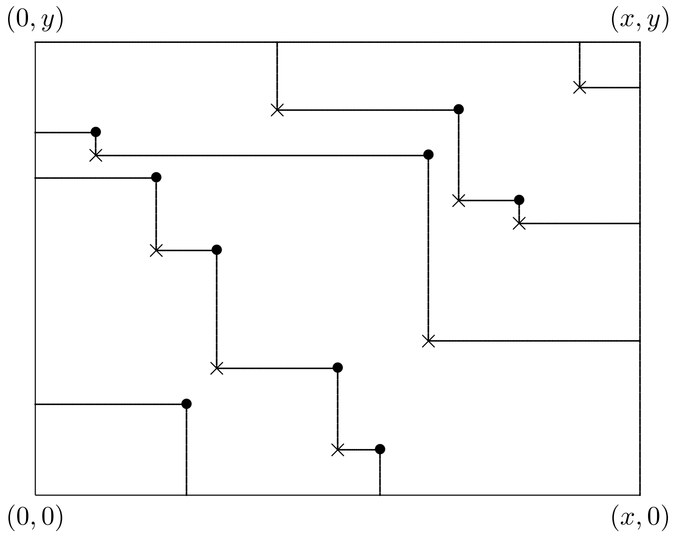


Figure 2: Space-time paths of the Hammersley's process, with sources and sinks.



(0, y)
(x, y)
(0, 0)
× α-point
• β-point
(x, 0)



(0, y)
(x, y)
(0, 0)
(x, 0)

2 Path of an isolated second class particle and local convergence of Hammersley's process

$t \mapsto L_\lambda(\cdot, t)$: Hammersley's process, considered as a 1-dimensional point process, developing in time t, generated by a Poisson process of sources on the positive x-axis of intensity λ, $\lambda > 0$, with Poisson sinks on the time axis of intensity $1/\lambda$, and a Poisson process of intensity 1 in $\mathbb{R}_+^2$.
The Poisson process on the x-axis, the Poisson process on the time axis, and the Poisson process in the plane are independent.
A *second class particle* jumps to the previous position of the particle that exits through the first sink at the time of exit, and successively jumps to the previous positions of particles directly to the right of it, at times where these particles jump to a position to the left of the second class particle (see Figure 3).

Theorem 2.1. *Let $t \mapsto L_\lambda(\cdot, t)$ be the stationary Hammersley process, defined above, with intensities λ and $1/\lambda$ on the x- and y-axis, respectively. Let X_t be the x-coordinate of an isolated second class particle w.r.t. L_λ at time t, located at the origin at time zero. Then*

$$t^{-1} X_t \xrightarrow{\text{a.s.}} 1/\lambda^2, \; t \to \infty. \tag{2.1}$$

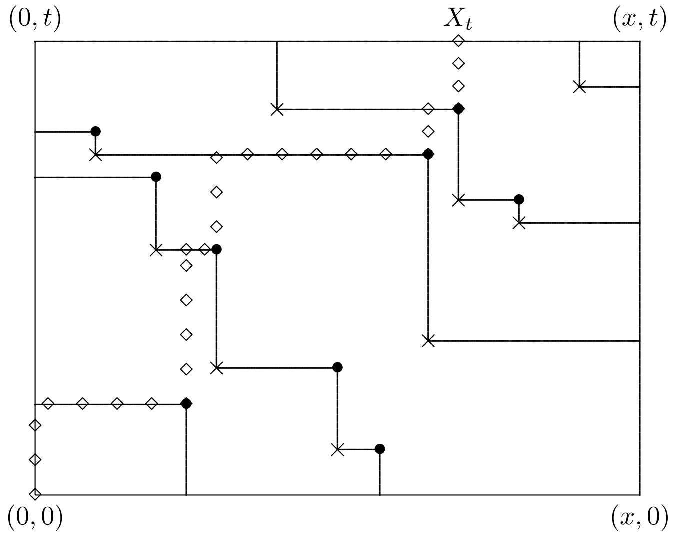


Figure 3: Path of isolated second class particle in the configuration of Figure 2

Some notation needed:

$\eta_t,\ t \geq 0$: stationary (1-dimensional) point process, originating from Poisson sources with intensity $\gamma > 0$ in $(0,\infty)$ at time 0, developing in time t according to Hammersley's process, with Poisson sinks of intensity $1/\gamma$ on the y-axis, and a Poisson point process of intensity 1 in the interior of the first quadrant.

$\sigma_t,\ t \geq 0$: stationary (1-dimensional) point process, coupled to $\eta_t,\ t \geq 0$, by using the same points in the first quadrant as used for η, and starting with a (δ/γ)–"thickening", $\delta > \gamma$, of the sources, obtained by adding independently a Poisson point process of intensity $\delta - \gamma$, and developing in time t according to Hammersley's process. The sinks on the y-axis are replaced by a γ/δ-thinned set, obtained by keeping each sink with probability γ/δ, independently for each sink.

$\eta_t[0,x]$: number of particles of η_t in the interval $[0,x]$ at time t.

$\sigma_t[0,x]$: number of particles of σ_t in the interval $[0,x]$ at time t.

Particles, escaping through a sink in the time interval $[0,t]$, are located at zero.

$\eta_t(0,x]$: number of particles of η_t in the interval $(0,x]$ at time t.

$\sigma_t(0,x]$: number of particles of σ_t in the interval $(0,x]$ at time t.

$t \mapsto \xi_t$: the process of *second class particles* of η w.r.t. σ: *the extra particles of σ_t w.r.t. η_t*.

Flux $F_\xi(x,t)$ of ξ through x at time t:

$$F_\xi(x,t) = \sigma_t[0,x] - \eta_t[0,x]. \tag{2.2}$$

$F_\xi(x,t)$ is equal to the number of second class particles in $(0,x]$ at time t minus the number of removed sinks in the segment $\{0\} \times [0,t]$ (through which space-time paths of second class particles start moving to the right).

Picture of the processes η and ξ: Figure 4.

In this particular case the process σ (inside the rectangle $[0,x] \times [0,t]$) is obtained from the process η by adding two sources at the locations $z_1(0)$ and $z_2(0)$ and removing a sink at height S_0. The crossings of horizontal lines of the space-time paths of the process σ are the unions of the crossings of (the same) horizontal lines of the space-time paths of the processes η and ξ.

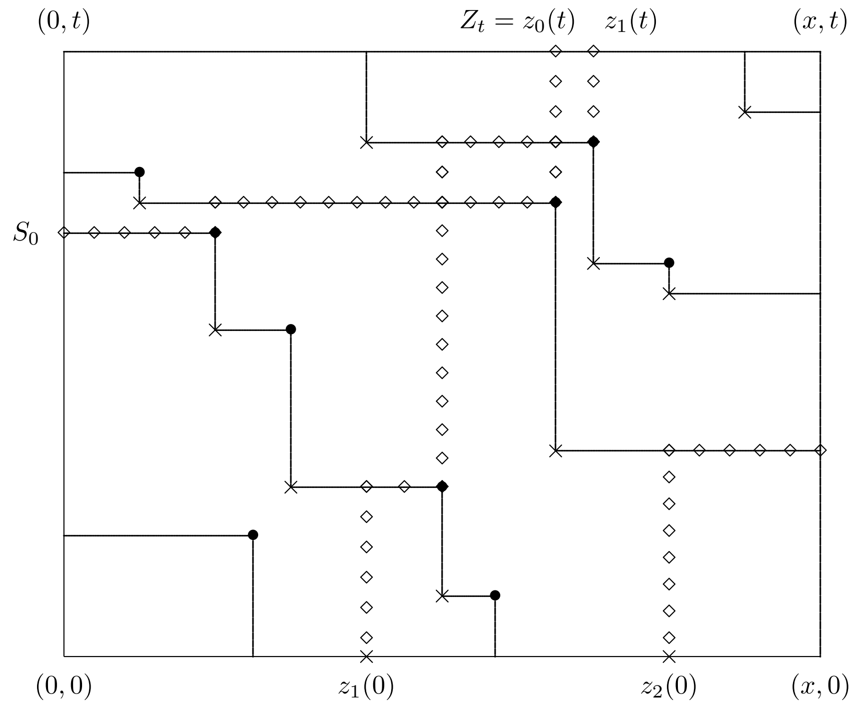


Figure 4: Processes η and ξ

Lemma 2.1.

(i) Let η and σ be defined as above, and let Z_t be the location of the second class particle at time t, for which the space-time path starts moving to the right through the smallest removed sink. Then

$$\lim_{t\to\infty}\frac{Z_t}{t}=\frac{1}{\gamma\delta}\,,\ \text{a.s.}$$

(ii) Similar statement for process η', developing from left to right:

$$\lim_{x\to\infty}\frac{Z'_x}{x}=\gamma\delta\,,\ \text{a.s.}$$

Proof.

(i): Let $x>0$. We have:

$$\lim_{n\to\infty}\frac{\eta_n[0,nx]}{n}=\frac{1}{\gamma}+x\gamma\,,\ \text{a.s.}\,,$$

since $\eta_n[0,nx]$ equals $\eta_n(0,nx]$ plus the number of sinks for the process η, contained in $\{0\}\times[0,n]$ (where n is a positive integer), and since $\eta_n(0,nx]$ and the number of sinks contained in $\{0\}\times[0,n]$ have Poisson distributions with parameters $nx\gamma$ and n/γ, respectively. Here we use the stationarity of the process η, implying that $\eta_n(0,nx]$ has a Poisson distribution with parameter $nx\gamma$. Note that, for each $\varepsilon>0$,

$$\sum_{n=1}^{\infty}P\left\{|\eta_n(0,nx]-nx\gamma|>n\varepsilon\right\}<\infty,$$

and hence, by the Borel-Cantelli Lemma:

$$P\left\{|\eta_n(0,nx]-nx\gamma|>n\varepsilon \text{ infinitely often}\right\}=0.$$

This implies the almost sure convergence of $\eta_n(0,nx]/n$ to $x\gamma$, as $n\to\infty$.

The almost sure convergence to $1/\gamma$ of the number of sinks for the process η, contained in $\{0\}\times[0,n]$, divided by n, is similarly a consequence of the fact that the number of sinks in $\{0\}\times[0,n]$ is Poisson(n/g).

In the same way:

$$\lim_{n\to\infty}\frac{\sigma_n[0,nx]}{n}=\frac{1}{\delta}+x\delta\,,\text{ a.s.}$$

Hence, by the definition of the flux ($F_\xi(x,t)=\sigma_t[0,x]-\eta_t[0,x]$), we get, a.s.,

$$\lim_{n\to\infty}\frac{F_\xi(nx,n)}{n}=\lim_{n\to\infty}\frac{\sigma_{nt}[0,nx]-\eta_{nt}[0,nx]}{n}=\frac{1}{\delta}-\frac{1}{\gamma}+x(\delta-\gamma)=-(\delta-\gamma)\left\{\frac{1}{\gamma\delta}-x\right\}. \tag{2.3}$$

This limit is negative for $0<x<1/(\gamma\delta)$ and positive for $x>1/(\gamma\delta)$.

We can number the particles of ξ according to their position at time 0, so that, for $i>0$, particle i is the ith second class particle to the right of the origin at time 0. We then let $z_i(t)$ be the position of the ith second class particle at time $t\ge 0$. For $i\le 0$, we let $z_i(t)$, $i=0,-1,-2,\dots$, be the second class particles at time t, for which the space-time paths leave the y-axis through the removed sinks $S_0,S_1,\dots$, respectively, ordering these removed sinks according to the height of their location on the y-axis; note that $Z_t=z_0(t)$ (see Figure 4).

Hence $F_\xi(x,t)$ has the representation:

$$F_\xi(x,t) = \#\{i>0 : z_i(t) \le x\} - \#\{i \le 0 : z_i(t) > x\}\,. \tag{2.4}$$

Note that second class particles $z_i(\cdot)$, $i \le 0$, starting their space-time path to the right at a removed sink in $\{0\}\times[0,t]$, and satisfying $z_i(t)\in[0,x]$, do not give a contribution to the flux $F_\xi(x,t)$, since they give a contribution to $\eta_t[0,x]$ as a particle of η_t, located at zero, and a contribution to $\sigma_t[0,x]$ as a particle of σ_t in the interval $(0,x]$. These two contributions cancel in $\sigma_t[0,x]-\eta_t[0,x]$.
It is also clear from the representation $F_\xi(x,t) = \#\{i>0 : z_i(t) \le x\} - \#\{i \le 0 : z_i(t) > x\}$ that, for fixed t, the flux $F_\xi(x,t)$ is nondecreasing in x.
The relation $F_\xi(x,t) = \#\{i>0 : z_i(t) \le x\} - \#\{i \le 0 : z_i(t) > x\}$ shows that $F_\xi(Z_n,n) = F_\xi(z_0(n),n)$ is equal to zero at each time n, and since $F_\xi(nx,n)$ is nondecreasing in x for fixed n, we get from

$$\lim_{n\to\infty}\frac{F_\xi(nx,n)}{n} = \lim_{n\to\infty}\frac{\sigma_{nt}[0,nx]-\eta_{nt}[0,nx]}{n} = \frac{1}{\delta}-\frac{1}{\gamma}+x(\delta-\gamma) = -(\delta-\gamma)\left\{\frac{1}{\gamma\delta}-x\right\},\ \text{a.s.},$$

that

$$\lim_{n\to\infty}\frac{Z_n}{n} = \frac{1}{\gamma\delta}\,,\ \text{a.s.}$$

But, since Z_t is nondecreasing in t, we then also have:

$$\lim_{t\to\infty}\frac{Z_t}{t} = \frac{1}{\gamma\delta}\,,\ \text{a.s.}$$

(ii): The result is obtained from part (i) by reflecting the processes w.r.t. the diagonal, and noting that the reflected processes have the same probabilistic behavior, but with the role of sources and sinks interchanged. The limit $1/(\gamma\delta)$ changes to $\gamma\delta$ because of the interchange of x- and y-coordinate. □

Proof of Theorem 2.1: We couple the process $t \mapsto (L_\lambda(\cdot,t), X_t)$ with the process $t \mapsto (\eta_t, \sigma_t)$, where the processes η and σ are defined as in part (i) of Lemma 2.1, and where $L_\lambda(\cdot,t) = \eta_t$ and $\delta > \gamma = \lambda$. Then $Z_t \le X_t$, for all $t \ge 0$, where Z_t is defined as in part (i) of Lemma 2.1.

This is seen in the following way. At time zero, we have $Z_0 = X_0 = 0$. Since the process σ is obtained from the process η by a thinning of the sinks and a "thickening" of the sources, and the space-time path of Z_t leaves the axis $\{0\} \times \mathbb{R}_+$ through the smallest *removed* sink, it will leave this axis at a time which is larger than or equal to the time the space-time path of X_t leaves the axis, since the space-time path of X_t will leave the axis through the smallest sink in the original set of sinks.

Note that since σ has less sinks and more sources:

$$\eta_t(0,x] \le \sigma_t(0,x],\ t \ge 0,\ x > 0. \tag{2.5}$$

This means that not only Z_t becomes positive at a time that is at least as large as the time that X_t becomes positive, but also moves to the right at a speed that is not faster than that of X_t. Also note that if Z_t jumps to a position $x > Z_{t-}$, an η-particle jumps over it from a position $x' \ge x$.

If $X_{t-} < x$ and $Z_{t-} \le X_{t-}$, X_t will jump to x'. Since $Z_t \le x'$, Z_t can never overtake X_t. Note that we can have $x' > x$ if several second class particle are next to each other, without a first class particle in between. In this case Z_t does not have to move to the position of the η particle, but can move to the position of the closest second class particle to the right of it.

Hence we have, with probability one:

$$\liminf_{t\to\infty} \frac{X_t}{t} \ge \lim_{t\to\infty} \frac{Z_t}{t} = \frac{1}{\gamma\delta} = \frac{1}{\delta\lambda}\,.$$

Since this is true for any $\delta > \lambda$, we get:

$$\liminf_{t\to\infty} \frac{X_t}{t} \ge \frac{1}{\lambda^2}\,.$$

For the reverse inequality, we switch the role of the sources and the sinks, and view Hammersley's process as developing from left to right. This time we add independently a Poisson point process of intensity $\delta^{-1} - \gamma^{-1}$ to the Poisson process of sinks of intensity γ^{-1}, and perform a δ/γ-thinning of the Poisson point process of sources of intensity γ on the x-axis, where $\gamma = \lambda$ and $0 < \delta < \gamma$, and use the process η' and σ', defined in part (ii) of Lemma 2.1. Note that η' has the same space-time paths as the process η, defined above. In the coupling we now consider L_λ as a process developing from left to right and take $L_\lambda(t,\cdot) = \eta'_t$.

Let X'_x be an isolated second class particle for the process running from left to right in the same way as X_t is an isolated second class particle for the process running upward. Trajectories of X and X' are shown in Figure 5.

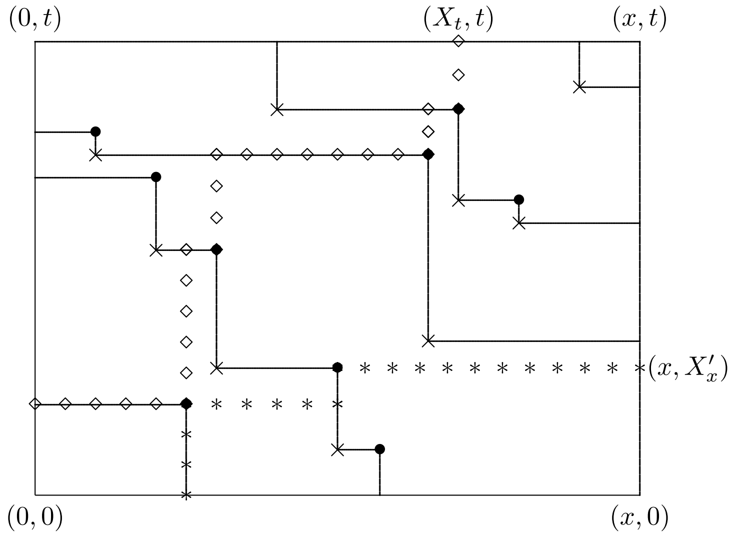


Figure 5: Trajectories of (X_t,t) and (x,X'_x)

We have:

$$X(X'(x)) \le x,\ x \ge 0, \tag{2.6}$$

writing temporarily $X'(x)$ instead of X'_x and $X(u)$ instead of X_u. Equation (2.6) is equivalent to the property of the process that the trajectory of (X_t,t) lies above the trajectory of (x,X'_x) (see also Figure 5). This follows by noting that if (X_t,t) hits a space-time path at a point North-West of the point where (x,X'_x) hits the same space-time path, this must also be true for the next space-time path, since the first trajectory moves up, and the second trajectory moves to the right.

By Lemma 2.1 and the argument above, now applied on the process moving from left to right, we get the relation

$$\liminf_{x\to\infty}\frac{X'_x}{x}\ge\lim_{x\to\infty}\frac{Z'_x}{x}=\delta\lambda\,, \tag{2.7}$$

with probability one. But the almost sure relation $\liminf_{x\to\infty}X'_x/x\ge\delta\lambda$ implies for the process $t\mapsto X_t$ the almost sure relation

$$\limsup_{t\to\infty}\frac{X_t}{t}\le 1/\left(\delta\lambda\right)\,, \tag{2.8}$$

since we get for each $\lambda'>1/\left(\delta\lambda\right)$, with probability one,

$$\limsup_{t\to\infty}\frac{X(t/\lambda')}{t/\lambda'}\le\limsup_{t\to\infty}\frac{X(X'(t))}{t/\lambda'}\le\lim_{t\to\infty}\frac{t}{t/\lambda'}=\lambda',$$

using (2.7) in the first inequality and (2.6) in the second inequality.

Since (2.8) is true for any $\delta<\lambda$, we get, with probability one,

$$\limsup_{t\to\infty}\frac{X_t}{t}\le\frac{1}{\lambda^2}.$$

□

Remark 2.1. The second class particle X'_x, introduced at the end of the proof of Theorem 2.1, plays the same role for Hammersley's process, running from left to right, as the second class particle X_t plays for Hammersley's process, running up. It therefore has to satisfy:

$$\lim_{x\to\infty} \frac{X'_x}{x} = \lambda^2, \text{ a.s.} \tag{2.9}$$

Note that we get an interchange of the x and t coordinate which leads to λ^2 in (2.10) instead of the $1/\lambda^2$ in (2.2), but that the line along which (x, X'_x) tends to ∞ is in fact the same as the line along which (X_t, t) tends to ∞.

The following lemma will allow us to show that Theorem 2.1 implies both the local convergence of Hammersley's process to a Poisson process and the relation $c = 2$ (which is the central result Theorem 5 on p. 204 in Aldous and Diaconis (1995)).

Lemma 2.2. *Let L_λ be the stationary Hammersley process, defined in Theorem 2.1. Furthermore, let L_λ^{-y} be the process, obtained from L_λ, by omitting the sinks on the y-axis, and let L_λ^{-x} be the process, obtained from L_λ, by omitting the sources on the x-axis. L_λ^{-y} is coupled to L_λ, by using the same point process in the interior of $\mathbb{R}_+^2$, and the same set of sources on the x-axis, and L_λ^{-x} is coupled to L_λ, by using the same point process in the interior of $\mathbb{R}_+^2$, and the same set of sinks on the y-axis. Then:*

(i) The processes L_λ and L_λ^{-y} have the same space-time paths below the space-time path $t \mapsto (X_t, t)$ of the isolated second class particle X_t for the process $t \mapsto L_\lambda(\cdot, t)$.

(ii) The processes L_λ and L_λ^{-x} have the same space-time paths above the space-time path $t \mapsto (t, X_t')$ of the isolated second class particle X_t' for the process $t \mapsto L_\lambda(t, \cdot)$, running from left to right.

Proof. (i). Omit the first sink at location y_1 on the y-axis. Then the path of L_λ leaving through $(0, y_1)$ is changed to a path traveling up through the β-point with y-coordinate y_1 to the right of $(0, y_1)$ until it hits the next path of the original process. At this level the path of the changed (by omitting the smallest sink) process is going to travel to the left, and the next path will go up (instead of to the left) through the closest β-point to the right. And so on. The "wave" through the β-points that is caused by leaving out the first sink is in fact the space-time path of the isolated second class particle X_t (see Figure 3).
We can now repeat the argument for the situation that arises by leaving out the second sink. This will lead to a "wave" through β-points that is going to travel North of the first wave that was caused by leaving out the first sink. This wave is the space-time path of an isolated second class particle in the new situation, where the first sink is removed. Below the first wave the space-time paths remain unchanged. The argument runs the same for all the remaining sinks.
(ii). The argument is completely similar, but now applies to the process running from left to right instead of up (see the end of the proof of Theorem 2.1). □

In the proof of Corollary 2.1 we will need the concept of a *weakly North-East path*, a concept also used in Baik and Rains (2000).

Definition 2.1. In the stationary version of Hammersley's process, a *weakly North-East path* is a North-East path that is allowed to pick up points from either the Poisson process on the x-axis or the Poisson process on the y-axis before going strictly North-East, picking up points from the Poisson point process in the interior $\mathbb{R}_+^2$. The *length of a weakly North-East path* from $(0,0)$ to (x,t) is the number of points of the Poisson processes on the axes and the interior of $\mathbb{R}_+^2$ on this path from $(0,0)$ and (x,t). A *strictly North-East path* is a path that has no vertical or horizontal pieces (and hence no points from the axes).

Note that the length of a longest weakly North-East path from $(0,0)$ to (x,t) in the stationary version of Hammersley's process is equal to the number of space-time paths intersecting $[0,x]\times[0,t]$, just as in the case of Hammersley's process without sources or sinks (in which case only strictly North-East paths are possible).

Corollary 2.1. (Theorem 5, Aldous and Diaconis (1995)) *Let L be Hammersley's process on $\mathbb{R}_+$, started from the empty configuration on the axes. Then,*

(i) For each fixed $a > 0$, the random particle configuration with counting process

$$y \mapsto L(t+y, at) - L(t, at),\ y \geq -t,$$

converges in distribution, as $t \to \infty$, to a homogeneous Poisson process on $\mathbb{R}$, with intensity $\sqrt{a}$.

(ii)

$$\lim_{t \to \infty} EL(t,t)/t = 2.$$

Proof. (i). Fix $a' > a$, and let, for $\lambda = \sqrt{a'}$, L_λ^{-y} be Hammersley's process, starting from Poisson sources of intensity λ on the positive x-axis, and running through an independent Poisson process of intensity 1 in the plane (without sinks). Then we get from Theorem 2.1 and Lemma 2.2 that the counting process $y \mapsto L_\lambda^{-y}(t+y, at) - L_\lambda^{-y}(t, at)$ converges in distribution to a Poisson process of intensity λ, since the process, restricted to a finite interval, lies with probability one at level t to the right of the space-time path of the isolated second class particle X_t, as $t \to \infty$.

If we couple the original Hammersley process and the process L_λ^{-y} via the same Poisson point process in the plane, we get that at any level the number of crossings of horizontal lines of the process L is contained in the set of crossings of these lines of the process L_λ^{-y}, since the latter process has sources on the x-axis and no sinks on the y-axis. Hence, for a finite collection of disjoint intervals $[a_i, b_i)$, $i = 1, \ldots, k$, and non-negative numbers $\theta_1, \ldots, \theta_k$, we obtain:

$$
\begin{aligned}
&E \exp\left\{-\sum_{i=1}^k \theta_i \left\{L(t+b_i, at) - L(t+a_i, at)\right\}\right\} \\
&\geq E \exp\left\{-\sum_{i=1}^k \theta_i \left\{L_\lambda^{-y}(t+b_i, at) - L_\lambda^{-y}(t+a_i, at)\right\}\right\}.
\end{aligned}
$$

But the right side converges by Theorem 2.1 and Lemma 2.2 to

$$
\exp\left\{-\sum_{i=1}^k \lambda(b_i - a_i)\left\{1 - e^{-\theta_i}\right\}\right\},
$$

so we get

$$
\liminf_{t\to\infty} E \exp\left\{-\sum_{i=1}^k \theta_i \left\{L(t+b_i, at) - L(t+a_i, at)\right\}\right\} \geq e^{-\sum_{i=1}^k \lambda(b_i - a_i)\left\{1-e^{-\theta_i}\right\}}. \tag{2.10}
$$

A similar argument, but now comparing the process L with a process L_λ^{-x}, having sinks of intensity $1/\lambda = 1/\sqrt{a'}$ on the y-axis (which can be considered to be "sources" for Hammersley's process, running from left to right), but no sources on the x-axis, shows

$$
\limsup_{t\to\infty} E \exp\left\{-\sum_{i=1}^k \theta_i \left\{L(t+b_i, at) - L(t+a_i, at)\right\}\right\} \leq e^{-\sum_{i=1}^k \lambda(b_i - a_i)\left\{1-e^{-\theta_i}\right\}}, \tag{2.11}
$$

for any $a' < a$, since in this case the crossings of horizontal lines of the process L are supersets of the crossings of these lines by the process L_{λ}^{-x}.

That the crossings of horizontal lines of the process L are supersets of the crossings of horizontal lines by the process L_{λ}^{-x} can be seen in the following way. Proceeding as in the proof of Lemma 2.2, we can, for the process L_{λ}, omit the sources one by one, starting with the smallest source. The omission of the smallest source will generate the path of a second class particle X_t', and the paths of L_{λ} will, at the interior of a vertical segment of the path of X_t', have an extra crossing of horizontal lines w.r.t. the paths of the process with the omitted source. On the other hand, the process with the omitted source will have extra crossings of $vertical$ lines, since some particles will make bigger jumps to the left. We can now repeat the argument by omitting the second source, which will lead to a further decrease of crossings of horizontal lines, etc.

Combining (2.10) and (2.11), we find:

$$\lim_{t\to\infty} E\exp\left\{-\sum_{i=1}^k \theta_i \left\{L(t+b_i,at)-L(t+a_i,at)\right\}\right\} = e^{-\sum_{i=1}^k (b_i-a_i)\sqrt{a}\left\{1-e^{-\theta_i}\right\}},$$

and the result follows.

(ii). Since the length of a longest strictly North-East path is always smaller than or equal to the length of a longest weakly North-East path, in the situation of a stationary process with Poisson sources on the positive x-axis and Poisson sinks on the positive y-axis, both with intensity 1, we must have, for each

$t > 0$,

$$EL(t,t)/t \le 2,$$

since the expected length of a longest weakly North-East path from $(0,0)$ to (t,t) is $2t$ for the stationary process.

The latter fact was proved in GROENEBOOM (2002), and arises from the simple observation that the length of a longest weakly North-East path from $(0,0)$ to (t,t) is equal to the total number of paths, crossing $\{0\} \times [0,t]$ and $[0,t] \times \{t\}$. Since the number of crossings of $\{0\} \times [0,t]$ has a Poisson(t) distribution by construction, and the number of crossings of $[0,t] \times \{t\}$ also has a Poisson(t) distribution, this time by the stationarity of the process L_λ, where $\lambda = 1$ in the present case, we get that the expectation of the total number of crossings of the left and upper edge is exactly $2t$.

To prove conversely that $\liminf_{t\to\infty} EL(t,t)/t \ge 2$, we first note that $L(t,t)$ is in fact the number of crossings of Hammersley's space-time paths with the line segment $[0,t] \times \{t\}$. Take a partition $0, t/k, 2t/k, \ldots, t$ of the interval $[0,t]$, for some integer $k > 0$. Then the crossings of the space-time paths of L of the segment $[(i-1)t/k, it/k] \times \{t\}$ *contain* the crossings of this line segment by the paths of a Hammersley process $L_{\lambda_i}^{-x}$ with sinks of intensity $1/\lambda_i = 1/\sqrt{a_i}, a_i < k/i$, on the y-axis, but no sources on the x-axis.

But, by Theorem 2.1 and Lemma 2.2, the crossings of the process $L_{\lambda_i}^{-x}$ with the segment $[(i-1)t/k, it/k] \times \{t\}$ belong, as $t \to \infty$, to the stationary part of the process with probability one, since $a_i < k/i$.

We now have:

$$\lim_{t\to\infty} t^{-1}E\left\{L_{\lambda_i}^{-x}(it/k,t) - L_{\lambda_i}^{-x}((i-1)t/k,t)\right\} = \frac{\lambda_i}{k}\,,$$

by uniform integrability of $t^{-1}L_{\lambda_i}^{-x}(\gamma t,t)$, $\gamma \in (0, i/k]$, $t \geq 0$, using (for example) the fact that the second moments are bounded above by the second moments of the corresponding stationary process with sources of intensity λ_i and sinks of intensity $1/\lambda_i$. Hence we get, by summing over the intervals of the partition:

$$\liminf_{t\to\infty} EL(t,t)/t \geq \frac{1}{k}\sum_{i=1}^{k}\sqrt{a_i}\,.$$

Letting $a_i \uparrow k/i$, we obtain (still for fixed k)

$$\liminf_{t\to\infty} EL(t,t)/t \geq \sum_{i=1}^{k} 1/\sqrt{ik} = 2(1+O(1/k)),$$

and (ii) follows by letting $k \to \infty$ in the latter relation. □

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