Empirical Processes Working Group
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Three Problems

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1 Problem 1

Example 1. (L_p deviations about the sample mean). Let $X, X_1, X_2, \ldots, X_n$ be i.i.d. P on R and let $\mathbb{P}_n$ denote the *empirical measure* of the X_i's:

Let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$, and, for $p \ge 1$ consider the L_p deviations about $\overline{X}_n$:

$$A_n(p) = \frac{1}{n}\sum_{i=1}^n |X_i - \overline{X}|^p = \mathbb{P}_n|X - \overline{X}_n|^p\,.$$

Questions:
(i) Does $A_n(p) \to_p E|X - E(X)|^p \equiv a(p)$?
(ii) Does $\sqrt{n}(A_n(p) - a(p)) \to_d N(0, V^2(p))$? And what is $V^2(p)$?
As will become clear, to answer question (i) we will proceed by showing that the class of functions $\mathcal{G}_\delta \equiv \{x \mapsto |x-t|^p : \ |t-\mu| \le \delta\}$ is a $P-$Glivenko-Cantelli class, and to answer question (ii) we will show that $\mathcal{G}_\delta$ is a $P-$Donsker class.

Example 1p. (L_p-deviations about the sample mean considered as a process in p. Suppose we want to study $A_n(p)$ as a stochastic process indexed by $p \in [a,b]$ for some $0 < a \le 1 \le b < \infty$. Can we prove that

$$\sup_{a \le p \le b} |A_n(p) - a(p)| \to_{a.s.} 0\,?$$

Can we prove that

$$\sqrt{n}(A_n - a) \Rightarrow \mathbb{A} \qquad \text{in} \quad D[a,b]$$

as a process in $p \in [a,b]$? This will require study of the empirical measure $\mathbb{P}_n$ and empirical process $\mathbb{G}_n$ indexed by the class of functions

$$\mathcal{F}_\delta = \{f_{t,p} : \ |t-\mu| \le \delta, a \le p \le b\}$$

where $f_{t,p}(x) = |x-t|^p$ for $x \in R$, $t \in R$, $p > 0$.

Example 1d. ($p-$th power of L_q deviations about the sample mean). Let $X, X_1, X_2, \ldots, X_n$ be i.i.d. P on R^d and let $\mathbb{P}_n$ denote the *empirical measure* of the X_i's:

Let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$, and, for $p, q \ge 1$ consider the deviations about $\overline{X}_n$ measured in the L_q-metric on R^d:

$$A_n(p,q) = \frac{1}{n}\sum_{i=1}^n \|X_i - \overline{X}\|_q^p = \mathbb{P}_n\|X - \overline{X}_n\|_q^p$$

where

$$\|x\|_q = (|x_1|^q + \cdots + |x_d|^q)^{1/q}\,.$$

Questions:
(i) Does $A_n(p) \rightarrow_p E\|X - E(X)\|_q^p \equiv a(p,q)$?
(ii) Does $\sqrt{n}(A_n(p,q) - a(p,q)) \rightarrow_d N(0, V^2(p,q))$? And what is $V^2(p,q)$?

2 Problem 2.

Example 2. Least L_p-estimates of location. Now suppose that we want to consider the measure of location corresponding to mimimum L_p-deviation:

$$\hat{\mu}_n(p) \equiv \text{argmin}_t\, \mathbb{P}_n|X - t|^p$$

for $1 \le p < \infty$. Of course $\hat{\mu}_n(2) = \overline{X}_n$ while $\hat{\mu}_n(1) =$ any median of $X_1, \ldots, X_n$. The asymptotic behavior of $\hat{\mu}_n(p)$ is well-known for $p = 1$ or $p = 2$, but for $p \ne 1, 2$ it is perhaps not so well-known. Consistency and asymptotic normality for any fixed p can be treated as a special case of the argmax (or argmin) continuous mapping theorem − which we will introduce as an important tool in chapter/lecture 2. The analysis in this case will again depend on various (Glivenko-Cantelli, Donsker) properties of the class of functions $\mathcal{F} = \{f_t(x) :\ t \in R\}$ with $f_t(x) = |x - t|^p$.

Example 2p. Least L_p estimates of location as a process in p. What can be said about the estimators $\hat{\mu}_n(p)$ considered as a process in p, say for $1 \le p \le b$ for some finite b? (Probably $b = 2$ would usually give the range of interest.)

Example 2d. Least p-th power of L_q- deviation estimates of location in R^d. Now supppose that $X_1, \ldots, X_n$ are i.i.d. P in R^d. Suppose that we want to consider the measure of location corresponding to mimimum L_q-deviation raised to the $p-$th power:

$$\hat{\mu}_n(p,q) \equiv \text{argmin}_t\, \mathbb{P}_n\|X - t\|_q^p$$

for $1 \le p, q < \infty$.

3 Problem 3.

Example 9.B. (Kendall's process). Suppose that $X \sim P$ on R^2 with distribution function H and marginal distributions F_1 and F_2. Then there is always a distribution function C on $[0,1]^2$ with uniform marginal distributions (a copula function) such that

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2)) .$$

Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with distribution function H, let $\epsilon = H(X)$, $\epsilon_i = H(X_i)$, and let Q denote the distribution function of the ϵ's:

$$Q(t) = P(H(X) \le t), \quad 0 \le t \le 1 .$$

A natural estimator of H is the empirical distribution function $\mathbb{H}_n$ of the X_i's, and hence the pseudo observations are

$$\hat{\epsilon}_{n,i} = \mathbb{H}_n(X_i) = \frac{1}{n} \#\{j \le n : X_j \le X_i\} ,$$

and the empirical distribution function of the $\hat{\epsilon}_{n,i}$'s is

$$\widehat{Q}_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{[0,t]}(\hat{\epsilon}_{n,i}) ,$$

the empirical measure of the $\hat{\epsilon}_{n,i}$'s indexed by the class of indicator functions $\mathcal{G} = \{1_{[0,t]} : t \in [0,1]\}$. In this case it is easily seen that Q and $\widehat{Q}_n$ do not depend on the marginal distributions F_1, F_2 of H, but only on the copula function C. This example has been considered in detail in Barbe, Genest, Ghoudi, and Rémillard (1996) and Ghoudi and Rémillard (1998). Questions:

(i) Does $Q_n(t) \to_p Q(t)$ uniformly in $t \in [0,1]$?
(ii) Does $\sqrt{n}(Q_n(t) - Q(t))) \Rightarrow \mathbb{Q}(t)$ for some Gaussian process $\mathbb{Q}$? [Yes! See Barbe, Genest, Ghoudi, and Rémillard (1996). But what is going on? Can the proof be simplified? What is the relationship to the class of "lower-layers"?]

References

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