# A note on the asymptotic distribution of Berk-Jones type statistics under the null hypothesis

Jon A. Wellner and Vladimir Koltchinskii

**Abstract.** Proofs are given of the limiting null distributions of the statistics of Berk and Jones (1979) and of Einmahl and McKeague (2002).

## 1. Introduction

Suppose that $X_1, \ldots, X_n$ are i.i.d. with distribution function $F$ on $R$ and we want to test

$$H : F = F_0 \quad \text{versus} \quad K : F \neq F_0 \tag{1.1}$$

where $F_0$ is continuous. Without loss of generality we can take $F_0(x) = (x \vee 0) \wedge 1$, the uniform distribution on $[0, 1]$. The test of Berk and Jones (1979) is defined in terms of the empirical distribution function $\mathbb{F}_n$ given, as usual, for $x \in R$ by

$$\mathbb{F}_n(x) = n^{-1} \sum_{i=1}^{n} 1_{[X_i \leq x]} \, .$$

For each fixed $x$ the random variable $n\mathbb{F}_n(x) \sim \text{Binomial}(n, F(x))$, and hence the likelihood ratio statistic for testing

$$H_x : F(x) = F_0(x) \quad \text{versus} \quad K_x : F(x) \neq F_0(x)$$

is given by

$$\begin{aligned}
\lambda_n(x) &= \frac{\sup_{F(x)} L_n(F(x))}{L_n(F_0(x))} = \frac{L_n(\mathbb{F}_n(x))}{L_n(F_0(x))} \\
&= \frac{\mathbb{F}_n(x)^{n\mathbb{F}_n(x)}(1 - \mathbb{F}_n(x))^{n(1-\mathbb{F}_n(x))}}{F_0(x)^{n\mathbb{F}_n(x)}(1 - F_0(x))^{n(1-\mathbb{F}_n(x))}} \\
&= \left(\frac{\mathbb{F}_n(x)}{F_0(x)}\right)^{n\mathbb{F}_n(x)} \left(\frac{1 - \mathbb{F}_n(x)}{1 - F_0(x)}\right)^{n(1-\mathbb{F}_n(x))} .
\end{aligned}$$

Received by the editors December 11, 2002.
1991 *Mathematics Subject Classification.* Primary 62G30; Secondary 62E20, 62E17.
*Key words and phrases.* Goodness of fit, null hypothesis, asymptotic distribution, centering.
We owe thanks to Bob Berk for encouragement to complete this study.

By defining

$$K(x,y) \equiv x \log\left(\frac{x}{y}\right) + (1-x)\log\left(\frac{1-x}{1-y}\right)$$

for $x, y \in [0,1]$, the log-likelihood ratio statistic is

$$\log \lambda_n(x) = nK(\mathbb{F}_n(x), F_0(x)) .$$

Note that $K(x,y)$ is the Kullback-Leibler "distance" between two Bernoulli distributions $P_x$ and $P_y$: $K(x,y) = K(P_x, P_y)$ where the "second $K$" is the Kullback-Leibler distance (or relative entropy)

$$K(P,Q) = E_P\{\log dP/dQ\} \qquad \text{if} \;\; P \ll Q$$

and $P_x(X = k) = x^k(1-x)^{1-k}$ for $k \in \{0,1\}$. It follows that $K(x,y) \ge 0$ with equality if and only if $x = y$ (which can also be proved directly).

The statistic $R_n$ studied by Berk and Jones (1979) is simply the supremum of these "pointwise likelihood ratio" test statistics:

$$R_n = \sup_{0 \le x \le 1} n^{-1} \log \lambda_n(x) = \sup_{0 \le x \le 1} K(\mathbb{F}_n(x), F_0(x)) .$$

Einmahl and McKeague (2003) propose an integral statistic $T_n$ defined by

$$T_n = 2\int_0^1 n^{-1} \log \lambda_n(x)\, dF_0(x) = 2\int_0^1 K(\mathbb{F}_n(x), F_0(x))\, dF_0(x) .$$

Einmahl and McKeague (2003) extend this integral type of test statistic to several other testing problems.

Our goal in this note is to give complete proofs of the following theorems of Berk and Jones (1979) and Einmahl and McKeague (2003). Note that neither of these papers provide proofs of these results since in both cases they are primarily concerned with other questions. Moreover, although the brief sketch of a proof of Theorem 1.1 given by Berk and Jones (1979) is heuristically on target, the first displayed formula of section 6 of Berk and Jones (1979) does not seem to be correct. The formula there asserts that

$$R_n = \frac{1}{2}\left\{\sup_{0<x<1} \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)}\right\}\left\{1 + O_p(n^{-1/2})\right\} ,$$

but our proof in section 2 suggests that only a weaker relationship holds.

**Theorem 1.1.** *Suppose that $F = F_0$; i.e. the null hypothesis $H$ holds. Then:*

$$nR_n - d_n \to_d Y_4 \sim E_v^4$$

*where $E_v^4(x) = \exp(-4\exp(-x)) = P(Y_4 \le x)$, and*

$$d_n = \log_2 n + \frac{1}{2}\log_3 n - \frac{1}{2}\log(4\pi)$$

*where $\log_2 n \equiv \log(\log n)$ and $\log_3 n \equiv \log(\log_2 n)$. In other words, for all $x \in R$*

$$\lim_{n\to\infty} P\left(nR_n - d_n \le x\right) = \exp\left(-4e^{-x}\right)$$

**Theorem 1.2.** *Suppose that $F = F_0$; i.e. the null hypothesis $H$ holds. Then:*

$$nT_n \to_d \int_0^1 \frac{\mathbb{U}^2(s)}{s(1-s)}\, ds \equiv A^2$$

*where $\mathbb{U}$ denotes a standard Brownian bridge process on $[0,1]$. In other words, for all $t \in R$ and*

$$\lim_{n\to\infty} P\left(nT_n \le t\right) = P\left(\int_0^1 \frac{\mathbb{U}^2(s)}{s(1-s)} ds \le t\right).$$

**Remark 1.** (Centering of $nR_n$). Set

$$\begin{aligned} c_n &= 2\log_2 n + (1/2)\log_3 n - (1/2)\log(4\pi)\,, \\ b_n &= (2\log_2 n)^{1/2}\,. \end{aligned}$$

As will be seen in the next section, the proof of Theorem 1.1 actually suggests that the appropriate centering of $nR_n$ is given by $c_n^2/(2b_n^2)$. Although

$$\frac{1}{2}\frac{c_n^2}{b_n^2} = \log_2 n + (1/2)\log_3 n - (1/2)\log(4\pi) + o(1) = d_n + o(1) \tag{1.2}$$

so the two centerings $c_n^2/(2b_n^2)$ and $d_n$ are asymptotically equivalent, simulations and comparison with the finite sample results of Owen (1995) suggest that the centering $c_n^2/(2b_n^2)$ yields a better approximation for finite sample sizes. We will provide evidence in support of this in Section 3.

**Remark 2.** Theorem 1.2 says that the asymptotic distribution of $nT_n$ under the null hypothesis is the same as that of the classical Anderson - Darling statistic; see e.g. Shorack and Wellner (1986), 148 and 224-227.

## 2. Proofs of Theorems 1.1 and 1.2.

We first prove theorem 1.1.

**Proof of Theorem 1.1:** First note that

$$\frac{\partial}{\partial x}K(x,y)\Big|_{x=y} = \log\left(\frac{x}{y}\right) - \log\left(\frac{1-x}{1-y}\right)\Big|_{x=y} = 0\,,$$

and

$$\frac{\partial^2}{\partial x^2}K(x,y) = \frac{1}{x} + \frac{1}{1-x} = \frac{1}{x(1-x)}\,.$$

Hence it follows that

$$\begin{aligned} K(x,y) &= K(y,y) + \frac{\partial}{\partial x}K(x,y)\Big|_{x=y}(y-x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}K(x,y)\Big|_{x=y^*}(y-x)^2 \\ &= 0 + 0 + \frac{1}{2}\frac{(y-x)^2}{y^*(1-y^*)} \\ &= \frac{1}{2}\frac{(y-x)^2}{y^*(1-y^*)} \end{aligned}$$

for some $y^*$ satisfying $|y^* - x| \le |y - x|$. This yields

$$K(\mathbb{F}_n(x), x) = \frac{1}{2} \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))} \tag{2.1}$$

for $0 < x < 1$ where $|\mathbb{F}_n^*(x) - x| \le |\mathbb{F}_n(x) - x|$; i.e. $x \le \mathbb{F}_n^*(x) \le \mathbb{F}_n(x)$ on the event $x \le \mathbb{F}_n(x)$ and $\mathbb{F}_n(x) \le \mathbb{F}_n^*(x) \le x$ on the event $\mathbb{F}_n(x) \le x$.

We can write (2.1) as

$$K(\mathbb{F}_n(x), x) = \frac{1}{2} \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \left\{ 1 + \frac{x(1-x)}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))} - 1 \right\}$$

where

$$\begin{aligned} |\mathrm{Rem}_n(x)| &\equiv \Big| \frac{x(1-x)}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))} - 1 \Big| \\ &= \Big| \frac{x - x^2 - (\mathbb{F}_n^*(x) - \mathbb{F}_n^*(x)^2)}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))} \Big| \\ &= \Big| \frac{(x - \mathbb{F}_n^*(x))(1 - (x + \mathbb{F}_n^*(x)))}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))} \Big| \\ &\le 3 \Big| \frac{(x - \mathbb{F}_n(x))}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))} \Big| . \end{aligned}$$

Fix $\delta \in (0, 1/2)$. Now for $x \in [\delta, 1-\delta]$, $\mathbb{F}_n(x) \in [\delta/2, 1 - \delta/2]$ a.s. for $n \ge N_\omega$, so

$$\sup_{\delta \le x \le 1-\delta} |\mathrm{Rem}_n(x)| \le \frac{3}{(\delta/2)(1-\delta/2)} \sup_{\delta \le x \le 1-\delta} |\mathbb{F}_n(x) - x| = O_p(n^{-1/2}) .$$

For $0 < x \le 1/2$ the function $g(x) = x(1-x)$ is $\nearrow$, so

$$g(\mathbb{F}_n^*(x)) \ge g(x) \wedge g(\mathbb{F}_n(x))$$

on the set $\{\mathbb{F}_n(x) < 1/2\}$. Since $P(\mathbb{F}_n(\delta) \ge 1/2) \to 0$, we get

$$\begin{aligned} \sup_{X_{(1)} \le x \le \delta} |\mathrm{Rem}_n(x)| &\le 3 \left( \sup_{X_{(1)} \le x \le \delta} \frac{|\mathbb{F}_n(x) - x|}{g(\mathbb{F}_n(x))} \vee \sup_{0 < x \le \delta} \frac{|\mathbb{F}_n(x) - x|}{g(x)} \right) \\ &= O_p(1) \end{aligned} \tag{2.2}$$

where $0 \le X_{(1)} \le \ldots \le X_{(n)} \le 1$ denote the order statistics of the sample. Here the $O_p(1)$ holds by virtue of Daniels (1945), Robbins (1954), and Chang (1955); see Theorem 2, Shorack and Wellner (1986), page 345, and Inequality 1, Shorack and Wellner (1986) page 415. Now note that $K(\mathbb{F}_n(x), x) = 0$ for $x < X_{(1)}$ and $x > X_{(n)}$. Therefore

$$\begin{aligned} R_n &= \sup_{0<x<1} K(\mathbb{F}_n(x), x) = \sup_{X_{(1)} \le x \le X_{(n)}} K(\mathbb{F}_n(x), x) \\ &= \frac{1}{2} \sup_{X_{(1)} \le x \le X_{(n)}} \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))} \end{aligned}$$

by (2.1). Fix $\delta \in (0, 1/2)$ and define

$$\begin{aligned} R_n(I) &= \frac{1}{2} \sup_{\delta \le x \le 1-\delta} \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))}, \\ R_n(II) &= \frac{1}{2} \sup_{X_{(1)} \le x \le \delta} \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))}, \\ R_n(III) &= \frac{1}{2} \sup_{1-\delta \le x \le X_{(n)}} \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))}. \end{aligned}$$

Thus $R_n = \max\{R_n(I), R_n(II), R_n(III)\}$. For $R_n(I)$,

$$\begin{aligned} R_n(I) &= \frac{1}{2} \sup_{\delta \le x \le 1-\delta} \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \left\{1 + O_p(n^{-1/2})\right\} \\ &\le \frac{1}{2} \sup_{\delta \le x \le 1-\delta} \left\{ \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \vee \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))} \right\} \left\{1 + O_p(n^{-1/2})\right\}. \end{aligned}$$

In the second region the argument above leading to (2.2) yields

$$\begin{aligned} R_n(II) &\le \frac{1}{2} \sup_{X_{(1)} \le x \le \delta} \left\{ \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \vee \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))} \right\} \\ &\ge \frac{1}{2} \sup_{X_{(1)} \le x \le \delta} \left\{ \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \wedge \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))} \right\}, \end{aligned}$$

and similarly for $R_n(III)$. It follows that

$$\begin{aligned} R_n &\le \frac{1}{2} \sup_{X_{(1)} \le x \le X_{(n)}} \left\{ \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \vee \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))} \right\} \\ &\qquad \times \left\{1 + O_p(n^{-1/2})\right\} \\ &= \frac{1}{2} \sup_{X_{(1)} \le x \le X_{(n)}} \left\{ \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \left(1 \vee \frac{x(1-x)}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))}\right) \right\} \\ &\qquad \times \left\{1 + O_p(n^{-1/2})\right\}, \end{aligned} \tag{2.3}$$

and, on the other hand,

$$\begin{aligned} R_n &\ge \frac{1}{2} \sup_{X_{(1)} \le x \le X_{(n)}} \left\{ \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \wedge \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))} \right\} \\ &\qquad \times \left\{1 + O_p(n^{-1/2})\right\} \\ &= \frac{1}{2} \sup_{X_{(1)} \le x \le X_{(n)}} \left\{ \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \left(1 \wedge \frac{x(1-x)}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))}\right) \right\} \\ &\qquad \times \left\{1 + O_p(n^{-1/2})\right\}. \end{aligned} \tag{2.4}$$

Now we break the suprema into the regions $[X_{(1)}, d_n]$, $[d_n, 1-d_n]$, and $[1-d_n, X_{(n)}]$ with $d_n = (\log n)^k/n$ for any $k \ge 1$. Then we have

$$n \sup_{X_{(1)} \le x \le d_n} \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} = o_p(b_n^2)$$

where $b_n = \sqrt{2 \log_2 n}$; see Shorack and Wellner (1986), (26), page 602. Moreover,

$$\sup_{X_{(1)} \le x \le d_n} \left| \frac{x(1-x)}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))} \right| = O_p(1)\,,$$

so

$$n \sup_{X_{(1)} \le x \le d_n} \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)} \left( 1 \,\#\, \frac{x(1-x)}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))} \right) = o_p(b_n^2) \tag{2.5}$$

for $\# = \wedge$ or $\# = \vee$, and similarly for the region $[1-d_n, X_{(n)}]$. On the other hand if we define

$$Z_n \equiv \sup_{d_n \le x \le 1 - d_n} \frac{\sqrt{n} |\mathbb{F}_n(x) - x|}{\sqrt{x(1-x)}}\,, \tag{2.6}$$

then, for $k \ge 5$

$$\frac{Z_n}{b_n} \to_p 1\,, \tag{2.7}$$

and

$$b_n Z_n - c_n \to_d E_v^4 \tag{2.8}$$

where $c_n = 2 \log_2 n + (1/2) \log_3 n - (1/2) \log(4\pi)$ (see e.g. Shorack and Wellner (1986), page 600, (16.1.20)) and (16.1.17)). Furthermore,

$$\left\| \frac{\mathbb{F}_n(x) - x}{x} \right\|_{d_n}^1 = O(r_n) \tag{2.9}$$

almost surely where

$$r_n^2 \equiv \frac{\log_2 n}{n d_n} = \frac{\log_2 n}{(\log n)^k} \to 0\,;$$

see Shorack and Wellner (1986), page 424, (4.5.10) and (4.5.11). It follows from (2.3), (2.4), (2.7), (2.8), and (2.9) that

$$\begin{aligned} nR_n &= \frac{1}{2} \left\{ \sup_{d_n \le x \le 1 - d_n} \frac{n(\mathbb{F}_n(x) - x)^2}{x(1-x)} (1 + O_p(r_n)) \bigvee o_p(b_n^2) \right\} \\ &\qquad \times \left\{ 1 + O_p(n^{-1/2}) \right\} \\ &= \frac{1}{2} \left\{ Z_n^2 \bigvee o_p(b_n^2) \right\} + o_p(1)\,. \end{aligned} \tag{2.10}$$

Hence we can write

$$\begin{aligned}\frac{1}{2}Z_n^2 &= \frac{1}{2}(Z_n - c_n/b_n)(Z_n + c_n/b_n) + \frac{1}{2}\frac{c_n^2}{b_n^2} \\ &= \frac{1}{2}b_n(Z_n - c_n/b_n)\frac{Z_n + c_n/b_n}{b_n} + \frac{1}{2}\frac{c_n^2}{b_n^2}\end{aligned}$$

It follows that

$$\begin{aligned}nR_n - \frac{1}{2}\frac{c_n^2}{b_n^2} &= b_n(Z_n - c_n/b_n)\frac{Z_n + c_n/b_n}{2b_n} \bigvee \left(o_p(b_n^2) - \frac{1}{2}\frac{c_n^2}{b_n^2}\right) + o_p(1) \\ &= b_n(Z_n - c_n/b_n)\frac{Z_n/b_n + c_n/b_n^2}{2} \bigvee (o_p(1) - 1/2)\, b_n^2 + o_p(1) \\ &\to_d Y_4 \frac{1+1}{2} \bigvee \{-\infty\} = Y_4\,; \qquad (2.11)\end{aligned}$$

here we used $c_n^2/b_n^2 \sim b_n^2$ in the second equality. Since

$$\frac{1}{2}\frac{c_n^2}{b_n^2} = \log_2 n + (1/2)\log_3 n - (1/2)\log(4\pi) + o(1) = d_n + o(1) \qquad (2.12)$$

this yields

$$P(nR_n - d_n \le x) \quad \to \quad \exp(-4\exp(-x))\,, \qquad (2.13)$$

and completes the proof of Theorem 1. Note that the centering $c_n^2/(2b_n^2)$ emerges naturally in the course of this proof. □

**Proof of Theorem 2.** Let $\alpha \in (1/2, 1)$, set $a_n = n^{-\alpha}$, and write

$$\begin{aligned}nT_n &= \left(\int_0^{a_n} + \int_{a_n}^{1-a_n} + \int_{1-a_n}^{1}\right) 2nK(\mathbb{F}_n(x), x)\, dx \\ &\equiv I_n + II_n + III_n\,.\end{aligned}$$

Now by (2.1) it follows that

$$II_n = \int_{a_n}^{1-a_n} \frac{n(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n^*(x)(1 - \mathbb{F}_n^*(x))}\, dx \to_d \int_0^1 \frac{\mathbb{U}^2(x)}{x(1-x)}\, dx$$

since

$$\left\| \frac{x(1-x)}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))} - 1 \right\|_{a_n}^{1-a_n} \to_p 0$$

by Wellner (1978) Theorem 0, page 77, and hence also with $\mathbb{F}_n$ replaced by $\mathbb{F}_n^*$, and

$$\int_{a_n}^{1-a_n} \frac{n(\mathbb{F}_n(x) - x)^2}{x(1-x)}\, dx \to_d \int_0^1 \frac{\mathbb{U}^2(x)}{x(1-x)}\, dx$$

as is well-know. Hence it remains only to show that $I_n \to_p 0$ (and then $III_n \to_p 0$ also by symmetry). To show this, fix $\epsilon > 0$ and choose $\lambda = \lambda_\epsilon$ so large that

$$P(\|\mathbb{F}_n(x)/x\|_0^1 > \lambda) = \lambda^{-1} < \epsilon\,.$$

On the event $\|\mathbb{F}_n(x)/x\|_0^1 \le \lambda$ we have

$$\begin{aligned} I_n &= \int_0^{a_n} 2nK(\mathbb{F}_n(x), x)\, dx \\ &= \int_0^{a_n} 2n\mathbb{F}_n(x) \log \frac{\mathbb{F}_n(x)}{x}\, dx + o(1) \qquad \text{a.s.} \\ &\le \int_0^{a_n} 2n\lambda x \log \lambda\, dx \\ &= \lambda \log \lambda\, n a_n^2 \to 0 \end{aligned}$$

since $\alpha > 1/2$. □

## 3. On the centering: finite sample approximations

Now we present some graphical evidence in favor of centering the statistic $nR_n$ of Berk and Jones (1979) by $c_n^2/(2b_n^2)$ rather than the asymptotically equivalent form given by $d_n$. Figures 1 and 2 below give empirical distributions of 5000 Monte Carlo replications of $nR_n - d_n$ for sample sizes $n = 100$ and $n = 1000$ respectively, together with the limit distribution function. From Figure 1 it is apparent that the distribution of $nR_n - d_n$ with $n = 100$ is shifted to the right from the limit distribution in the middle of its range. From Figure 2 we see that this continues to be the case for $n = 1000$, although the shift is somewhat less. Figures 3 and 4 give empirical distributions of 5000 Monte Carlo replications of $nR_n - c_n^2/(2b_n^2)$ for samples sizes $n = 100$ and $n = 1000$ respectively, together with the limit distribution function. It is clear that the distribution of $nR_n - c_n^2/(2b_n^2)$ nearly coincides with the limit distribution in the middle of its range. Furthermore, Figure 4 shows improved agreement with the limit distribution using the centering $c_n^2/(2b_n^2)$ for $n = 1000$.

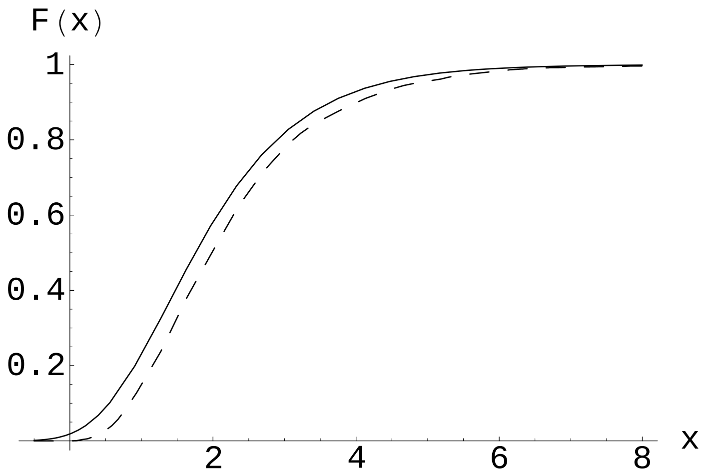


Figure 1: solid line: limit distribution $F(x) = \exp(-4\exp(-x))$; dashed line: empirical distribution function of 5000 replications of $nR_n - d_n$, $n = 100$.

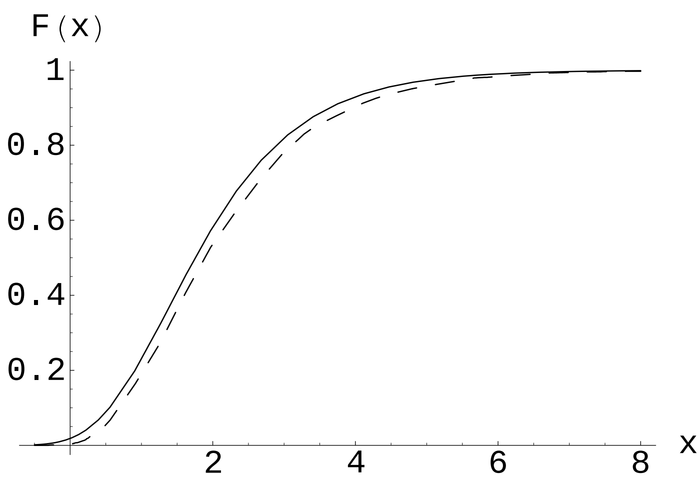


Figure 2: solid line: limit distribution $F(x) = \exp(-4\exp(-x))$; dashed line: empirical distribution function of 5000 replications of $nR_n - d_n$, $n = 1000$.

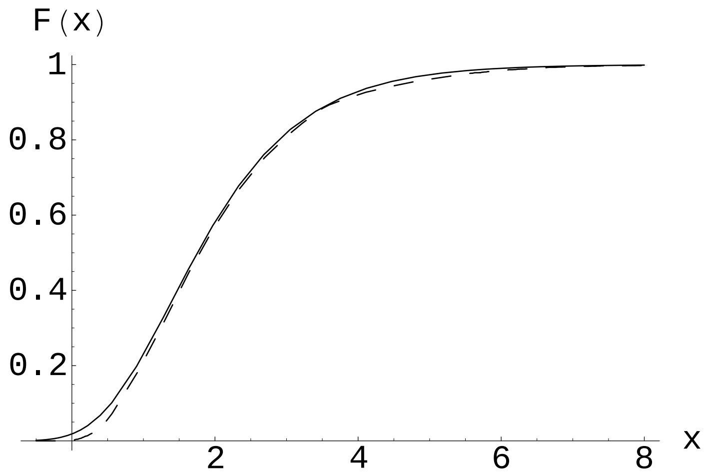


Figure 3: solid line: limit distribution $F(x) = \exp(-4\exp(-x))$; dashed line: empirical distribution function of 5000 replications of $nR_n - c_n^2/(2b_n^2)$, $n = 100$.

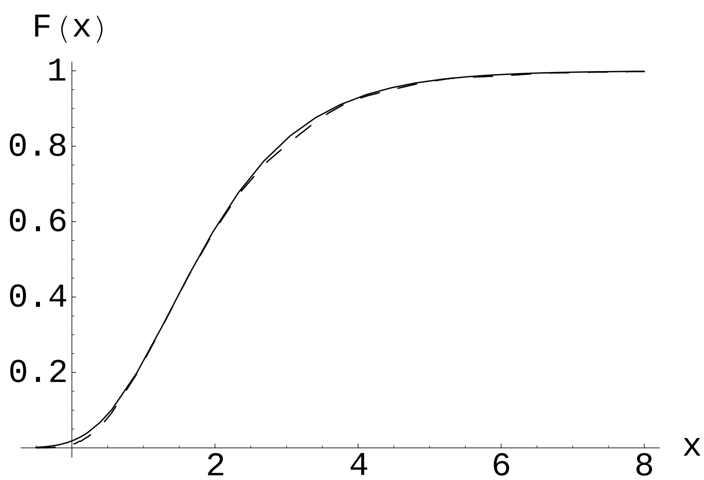


Figure 4: solid line: limit distribution $F(x) = \exp(-4\exp(-x))$; dashed line: empirical distribution function of 5000 replications of $nR_n - c_n^2/(2b_n^2)$, $n = 1000$.

Owen (1995) used the recursions of Noe (1972) to find finite sample formulas for the quantiles of $nR_n$. For example, Owen (1995) reports that approximate .95

and .99 quantiles of $nR_n$ for $1 < n \leq 1000$ are given as follows:

$$\lambda(n, .95) \approx \begin{cases} 3.0123 + .4835 \log n - .00957(\log n)^2 - .001488(\log n)^3\,, & 1 < n \leq 100 \\ 3.0806 + .4894 \log n - .02086(\log n)^2\,, & 100 < n \leq 1000\,, \end{cases}$$

while

$$(3.1)\lambda(n, .99) \approx \begin{cases} 4.626 + .541 \log n - .0242(\log n)^2\,, & 1 < n \leq 100\,, \\ 4.710 + .512 \log n - .0219(\log n)^2\,, & 100 < n \leq 1000. \end{cases}$$

The signs in (3.1) are from Owen (2001), page 159, Table 7.1, and differ by a sign from Owen (1995) formulas (12) and (13); it seems clear that this is a typo in Owen (1995).

Figures 5 and 6 give plots of these approximations (in blue) from Owen (1995) together with the corresponding quantiles resulting from Theorem 1 with the two asymptotically equivalent centerings $c_n^2/(2b_n^2)$ and $d_n$. Figure 5 shows that centering by $c_n^2/(2b_n^2)$ gives excellent correspondence with Owen's formula for $1 - \alpha = .95$, and we therefore propose using the resulting formula, namely

$$\lambda(n, .95; c_n^2/(2b_n^2)) = \frac{c_n^2}{2b_n^2} - \log\{(1/4)\log(1/(1 - .05))\}$$

rather than

$$\lambda(n, .95; d_n) = d_n - \log\{(1/4)\log(1/(1 - .05))\}$$

for $n \geq 800$ (and certainly for $n \geq 1000$). Figure 6 gives a less clear picture for $1 - \alpha = .99$. Although the approximation using the centering $c_n^2/(2b_n^2)$ is closer to Owen's finite sample formula, there is still not a clear agreement for $n \geq 1000$, so further work is needed here.

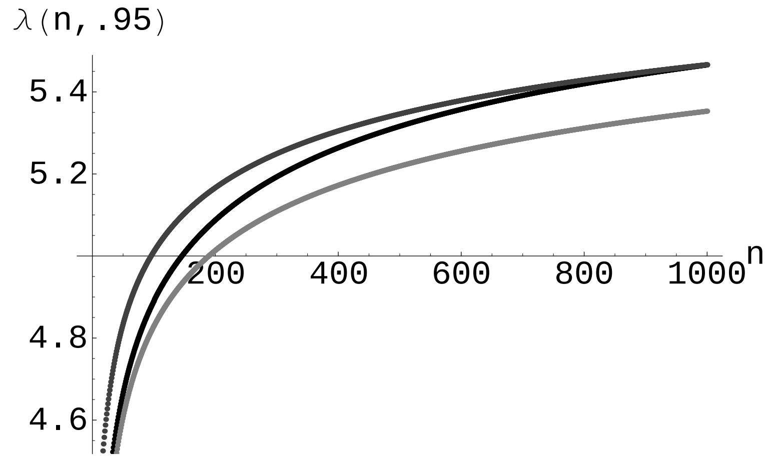


Figure 5. Black: Owen's finite-sample quantiles, $\lambda(n, .95)$.
Medium Gray: approximte quantiles $\lambda(n, .95; c_n^2/(2b_n^2))$
Light Gray: approximate quantiles $\lambda(n, .95; d_n)$

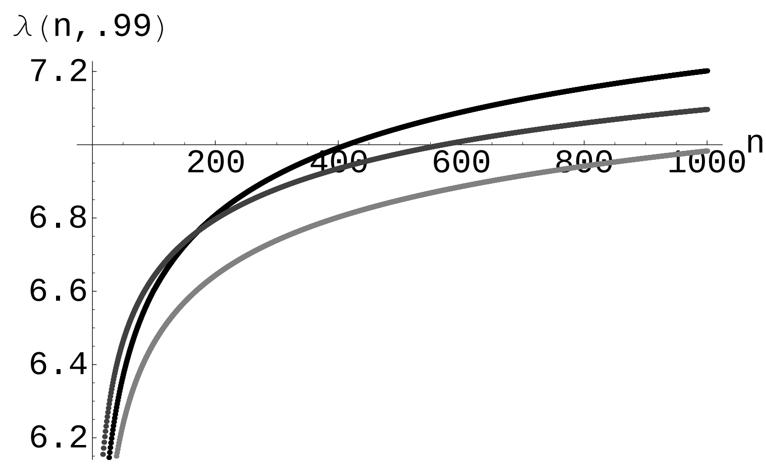


Figure 6. Black: Owen's finite-sample quantiles, $\lambda(n, .99)$.
Medium Gray: approximte quantiles $\lambda(n, .99; c_n^2/(2b_n^2))$
Light Gray: approximate quantiles $\lambda(n, .99; d_n)$

**Acknowledgements:** We owe thanks to Bob Berk for encouragement to complete this study.

## References


[1] Berk, R. H. and Jones, D. H. (1978). Relatively optimal combinations of test statistics. *Scand. Journal of Statistics* **5**, 158 - 162.

[2] Berk, R. H. and Jones, D. H. (1979). Goodness-of-fit statistics that dominate the Kolmogorov statistics. *Z. Wahrsch. verw. Gebiete* **47**, 47 - 59.

[3] Chang, Li-Chien (1955). On the ratio of the empirical distribution to the theoretical distribution function. *Acata Math. Sinica* **5**, 347 - 368. [English Tranlation in *Selected Tranls. Math. Statist. Prob.* **4**, 17 - 38 (1964).]

[4] Daniels, H. E. (1945). The statistical theory of the strength of bundles of thread. *Proc. Roy. Soc. London, Ser. A* **183**, 4-5 - 435.

[5] Darling, D. A. and Erdös, P. (1956). A limit theorem for the maximum of normalized sums of independent random variables. *Duke Math. J.* **23**, 143-155.

[6] Donoho, D. and Jin, J. (2002). Higher criticism for detecting sparse heterogeneous mixtures. Technical Report 2002-12, Department of Statistics, Stanford University.

[7] Eicker, F. (1979). The asymptotic distribution of the suprema of the standardized empirical process. *Ann. Statist.* **7**, 116 - 138.

[8] Einmahl, J.H.J. and McKeague, I. (2003). Empirical likelihood based hypothesis testing. *Bernoulli* **9**, 267 - 290.

[9] Groeneboom, P. and Shorack, G. R. (1981). Large deviations of goodness of fit statistics and linear combinations of order statistics. *Ann. Probab.* **9**, 971 - 987.

[10] Jaeschke, D. (1979). The asymptotic distribution of the supremum of the standardized empirical distibution function on subintervals. *Ann. Statist.* **7**, 108 - 115.

[11] Mason, D. M. and Schuenemeyer, J. H. (1983). A modified Kolmogorov-Smirnov test sensitive to tail alternatives. *Ann. Statist.* **11**, 933-946.

[12] Noe, M. (1972). The calculation of distributions of two-sided Kolmogorov-Smirnov type statistics. *Ann. Math. Statist.* **43**, 58 - 64.

[13] Owen, A. B. (1995). Nonparametric likelihood confidence bands for a distribution function. *J. Amer. Statist. Assoc.* **90**, 516 - 521.

[14] Owen, A. B. (2001). *Empirical Likelihood.* Chapman and Hall/CRC, Boca Raton.

[15] Robbins, H. (1954). A one-sided confidence interval for an unknown distribution function. *Ann. Math. Statist.* **25**, 409.

[16] Shorack, G. R. and Wellner, J. A. (1978). Linear bounds on the empirical distribution function. *Ann. Prob.* **6**, 349 - 353.

[17] Shorack, G. R. and Wellner, J. A. (1986). *Empirical Processes with Applications to Statistics.* Wiley, New York.

[18] Wellner, J. A. (1978). Limit theorems for the ratio of the empirical distribution function to the true distribution function. *Z. Wahrsch. verw. Geb.* **45**, 73 - 88.

**Acknowledgment**

We owe thanks to Bob Berk for encouragement to complete this study.

University of Washington, Statistics, Box 354322, Seattle, Washington 98195-4322, U.S.A.
*E-mail address*: `jaw@stat.washington.edu`

Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131-1141, U.S.A.
*E-mail address*: `vlad@math.unm.edu`