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MONTE CARLO OF TWO-DIMENSIONAL BROWNIAN SHEETS

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1. <u>Introduction</u>. A Brownian sheet $Z(\underset{\sim}{t})$ with an r-dimensional parameter set is a mean zero Gaussian process defined on the positive orthant of r-dimensional Euclidean space, R_r^+, having covariance $EZ(\underset{\sim}{s})Z(\underset{\sim}{t}) = |\underset{\sim}{s} \wedge \underset{\sim}{t}|$ (the minimum is taken coordinate by coordinate and $|\underset{\sim}{t}| \equiv t_1 \ldots t_r$). Such processes arise, for example, as the (weak) limit of "partial sum processes" on an r-dimensional grid, N_r^+. These processes have received increasing attention (see Pyke [3] for a recent review of the status of research), but few researchers have "seen" a Brownian sheet, and knowledge concerning the fluctuation theory of these processes is extremely scant. For example, the distributions of $M_r \equiv \sup_{\underset{\sim}{0} \le \underset{\sim}{t} \le \underset{\sim}{1}} Z(\underset{\sim}{t})$ and $T_r \equiv \lambda\{\underset{\sim}{t} \le \underset{\sim}{1}: Z(\underset{\sim}{t}) > 0\}$ (λ denotes Lebesgue measure on R_r^+) are unknown for $r \ge 2$. (For $r=1$ the distributions are, of course, well known: M_1 has a half-normal distribution and T_1 has the arcsin distribution.)

Here we present the results of a Monte Carlo experiment in which approximations to Brownian sheets on R_2^+ were obtained as partial sums of Normal (0,1) rv's. The aims of this experiment were twofold: first, to plot several realizations of the approximating partial sum process so that we could "see" several Brownian sheets; second, to learn about the distributions of M_2 and T_2 by examining the empirical distribution functions of the corresponding rv's defined on approximating partial sum processes. A plot of the partial sum process itself is shown in Section 2 and may give some idea of the appearance of a Brownian sheet. In Section 3 we present the empirical distributions which estimate the distributions of M_2 and T_2. We also show that the arcsin law does not hold for $r \geq 2$; in fact, $T_r \overset{P}{\to} 1/2$ as $r \to \infty$. Finally, for the sake of completeness, the Appendix lists the computer programs used to generate the partial sum processes and perform the computation of empirical distributions.

The first simulations on Brownian sheets known to this author were carried out at the University of Minnesota in the summer of 1973, as communicated to me through Professor John B. Walsh. The print-outs in this case were "pictures" of the regions where a Brownian sheet is positive. Similar pictures were later obtained by Professor Pyke at the University of Washington. A question raised by Professor Walsh concerns the apparent rectangularity of the level lines present in some of the ω's.

2. A picture of a Brownian sheet on R_2^+. Figure 1 is a "three dimensional" plot of a sample path (one ω in the sample space, of a partial sum process $S_{\underset{\sim}{k}}$, $\underset{\sim}{1} \leq \underset{\sim}{k} \leq \underset{\sim}{m}$ obtained by summing independent Normal

(0,1) rv's, $X_{\underset{\sim}{j}}$'s, as follows:

$$S_{\underset{\sim}{k}} = \sum_{\underset{\sim}{j} \le \underset{\sim}{k}} X_{\underset{\sim}{j}}, \qquad \underset{\sim}{1} \le \underset{\sim}{k} \le \underset{\sim}{m}. \tag{1}$$

The grid is 50 by 50 ($\underset{\sim}{m} = (50,50)$) and the sheet is viewed from the point $(x,y,z) = (-833,-833,+833)$ in grid units. (Or from (-100,-100,+100) in inches with the 50 by 50 grid on a six inch square.) The origin is not plotted, so the entire sheet is seen relative to the value of $X_{\underset{\sim}{1}} = S_{\underset{\sim}{1}}$ and not $S_{\underset{\sim}{0}} = 0$. Although the sheet appears to be entirely positive in this view, there are several valleys and "holes" which are not visible from this viewing position, but can be seen clearly from other angles. For a 25 by 25 grid, several sheets were viewed from all eight corners.

3. Empirical distributions of M_2 and T_2. In terms of the partial sums $S_{\underset{\sim}{k}}$, the random variables corresponding to M_r and T_r are

$$M_r^d = \max_{\underset{\sim}{1} \le \underset{\sim}{k} \le \underset{\sim}{m}} S_{\underset{\sim}{k}} / |\underset{\sim}{m}|^{\frac{1}{2}} \tag{2}$$

and

$$T_r^d = \#\{\underset{\sim}{k} \le \underset{\sim}{m}: S_{\underset{\sim}{k}} > 0\} / |\underset{\sim}{m}|. \tag{3}$$

Figures 2 through 5 present empirical distributions of M_2^d and T_2^d based on a 50 by 50 grid ($\underset{\sim}{m} = (50,50)$)

and two independent samples, each of size n = 400. (The empirical df of $|\underset{\sim}{m}|^{\frac{1}{2}}M_2^d = \max S_{\underset{\sim}{k}}$ is plotted in Figures 4 and 5 rather than M_2^d itself.) Figures 6 and 7 show the locations of the maxima ($\underset{\sim}{K}$ such that $M_2^d = S_{\underset{\sim}{K}}$) on the 50 by 50 grid for the same two samples. Each sample of 400 sheets, involving the generation of 10^6 normal random variables, required approximately 6 minutes of central processor time on the CDC 6400 at the University of Washington. The program used is given in an Appendix.

With only minor modifications the program could be used to obtain more detailed information concerning the distributions of functionals of the process Z with r = 2. For example, in view of the intractability of the distribution of M_2 (and its "tied down" analogue which is of importance for statistical applications), accurate estimates of several upper percent points of the distribution of M_2 would be of interest. Accurate estimates of quantiles of M_2 could be obtained using the present program with increased sample size, and estimates of quantiles for the "tied down" version of M_2 could be obtained by suitably transforming the partial sum process.

The empirical distributions of Figures 2 and 3 suggested that T_2 might have a uniform (0,1) distribution which could be made compatible with the 1-dimensional arcsin distribution through a Beta family of distributions. This null hypothesis that the distribution was uniform (0,1) was then tested using the two-sided Kolmogorov test. The hypothesis was accepted! However, the distribution is not uniform (0,1) as may be seen by calculating the variance of T_2: using Sheppard's formula ([1], page 125) and doing some definite integrals one obtains

$$\text{var } T_2 = (2\pi)^{-1}\int_0^1 \int_0^1 \arcsin\, (x_1x_2)^{\frac{1}{2}}dx_1dx_2$$

$$= (1/4)\ (1 - \log 2).$$

On the other hand, Figures 2 and 3 do suggest that the density function of T_2 is nearly flat, a dramatic change from the bowl-shaped arcsin density of T_1, and leads one to the natural conjecture that the densities of T_r "turn around" as r increases and become concentrated about $\frac{1}{2}$. The following proposition confirms this guess.

Proposition. $T_r \xrightarrow{P} \frac{1}{2}$ as $r \to \infty$. (In fact, var $T_r \le (1/4)\ (2/3)^r$.)

Proof. Using Sheppard's formula and the inequality $\arcsin (t) \le (\pi/2)t$ for $0 \le t \le 1$ we obtain

$$\text{var } T_r = (2\pi)^{-1}\int_{\underset{\sim}{0}}^{\underset{\sim}{1}} \arcsin\, (|\underset{\sim}{x}|^{\frac{1}{2}})\ |d\underset{\sim}{x}|$$

$$\le (1/4)\int_{\underset{\sim}{0}}^{\underset{\sim}{1}} |\underset{\sim}{x}|^{\frac{1}{2}}|d\underset{\sim}{x}|$$

$$= (1/4)\ (2/3)^r$$

$$\to 0 \text{ as } r \to \infty$$

and the conclusion follows since $ET_r = \frac{1}{2}$ for all r.

Questions: (1) Does T_r, properly normalized, have a non-degenerate limiting distribution as $r \to \infty$? (2) What happens to the distribution of M_r

as $r \to \infty$? (Possibly an extreme value distribution?)
(3) The central question remains: Is there a "fluctuation theory" for r-dimensional sheets and arrays of which the known results for $r = 1$ (see [2], page 419) are a special case?

Acknowledgment. I wish to express my appreciation to Professor Ron Pyke for several stimulating discussions concerning the subject of this paper and for his encouragement of the work presented here. Thanks are also due to Gary Morishima who helped ease the problems of communicating with the computer.

REFERENCES

[1] Bickel, P.J. (1971). Mathematical Statistics. (Preliminary edition, Part I). Holden-Day, San Francisco.

[2] Feller, W. (1971). An Introduction to Probability Theory and Its Applications, vol. II, 2nd edition. Wiley, New York.

[3] Pyke, R. (1973). Partial Sums of Matrix Arrays and Brownian Sheets. In Stochastic Analysis, D.G. Kendall and E.F. Harding, editors. Wiley, New York.

Figure 1. A Brownian sheet on R_2^+.

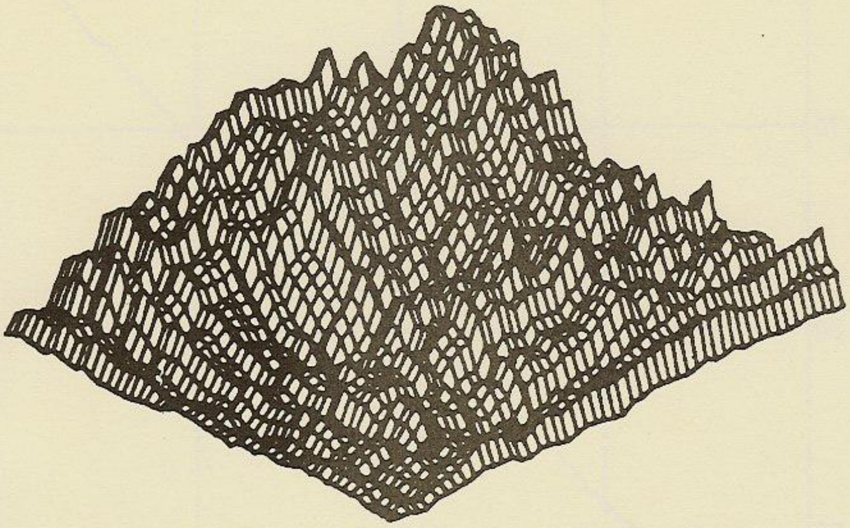


Figure 2. Empirical distribution of T_2^d, n = 400, sample 1.

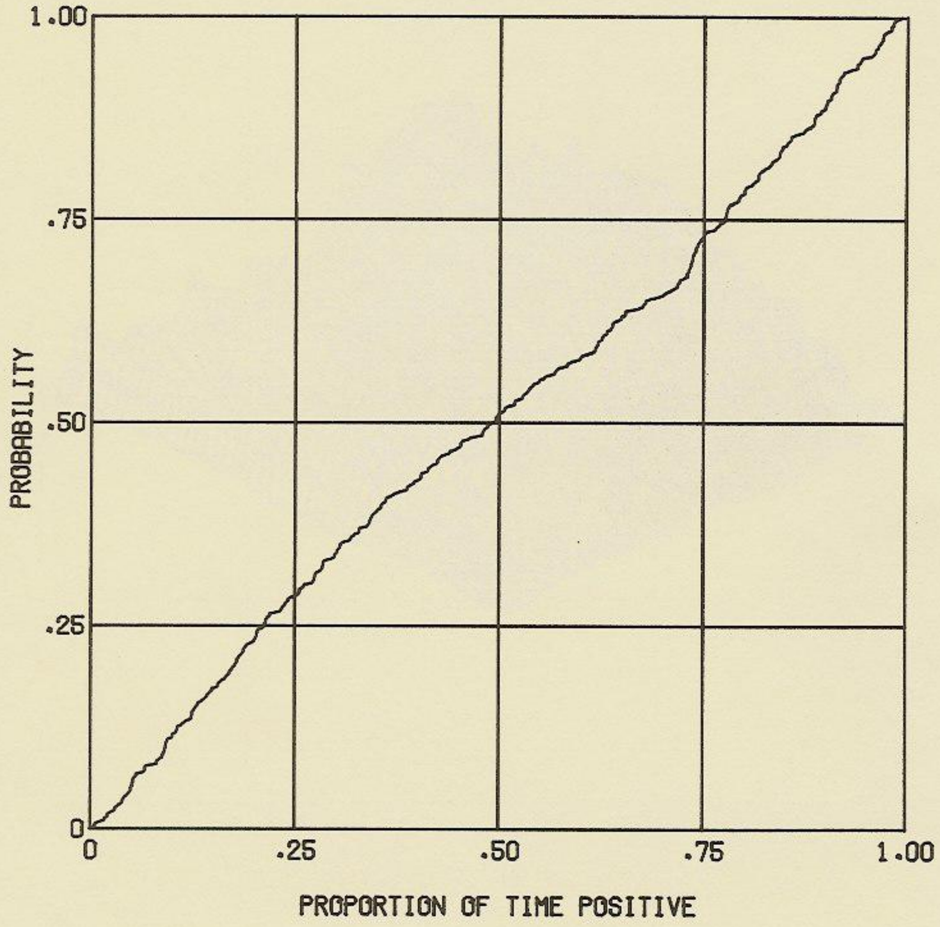


Figure 3. Empirical distribution of T_2^d, n = 400, sample 2.

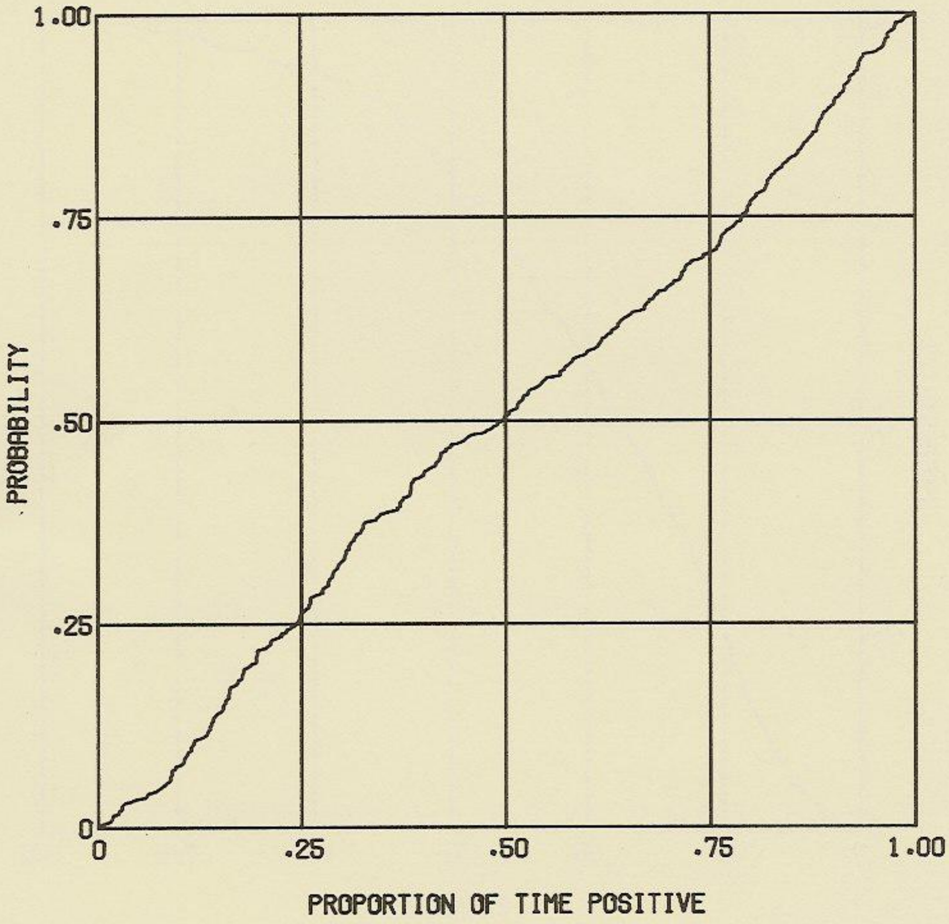


Figure 4. Empirical distribution of M_2^d, n = 400, sample 1.

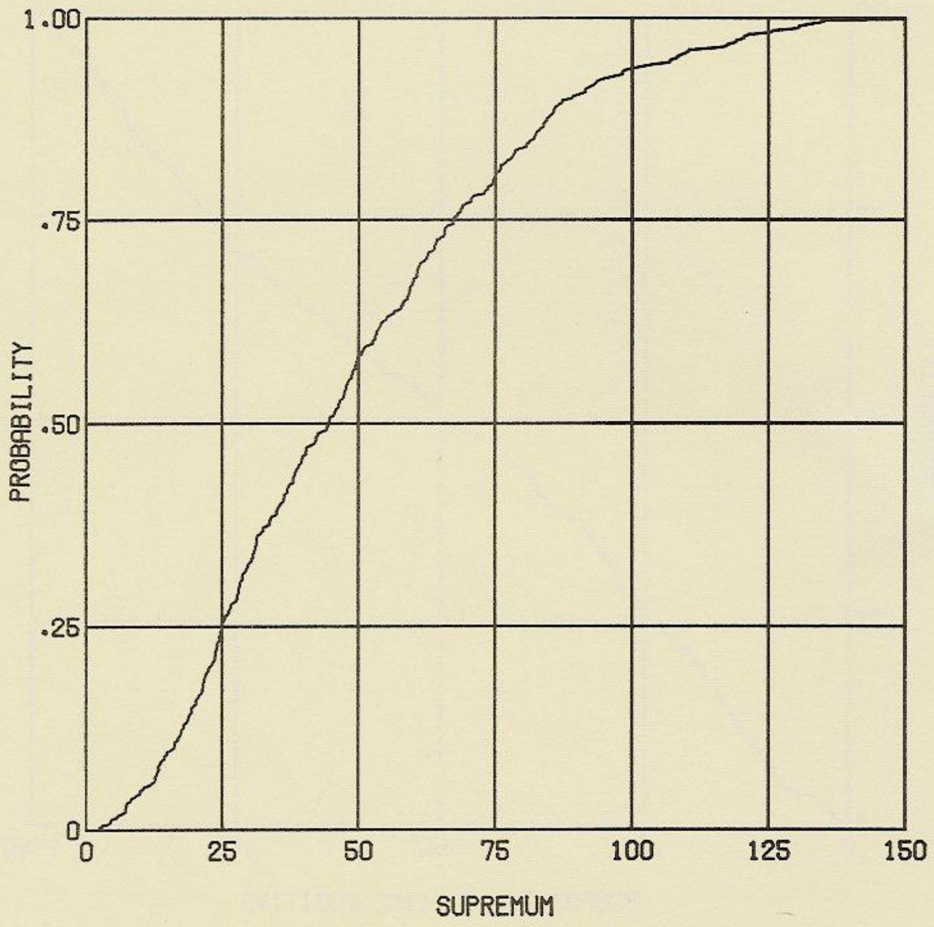


Figure 5. Empirical distribution of M_2^d, n = 400, sample 2.

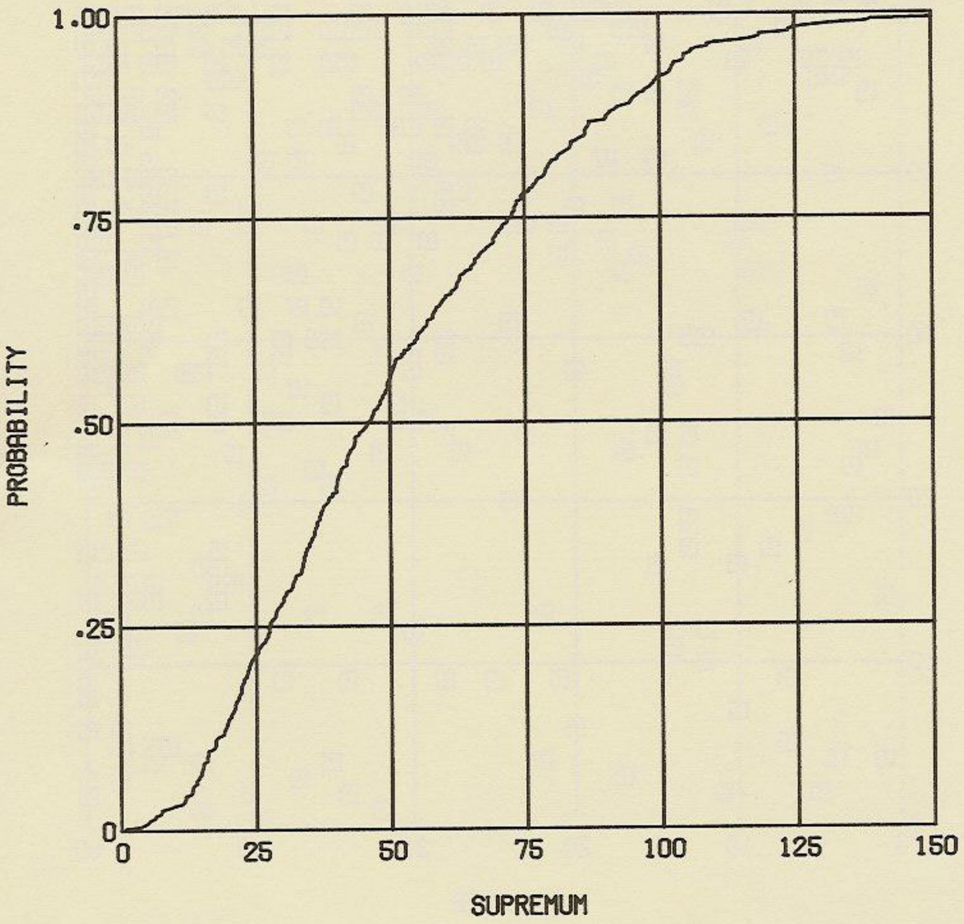


Figure 6. Location of the maxima, $\underset{\sim}{K}$, n = 400, sample 1.

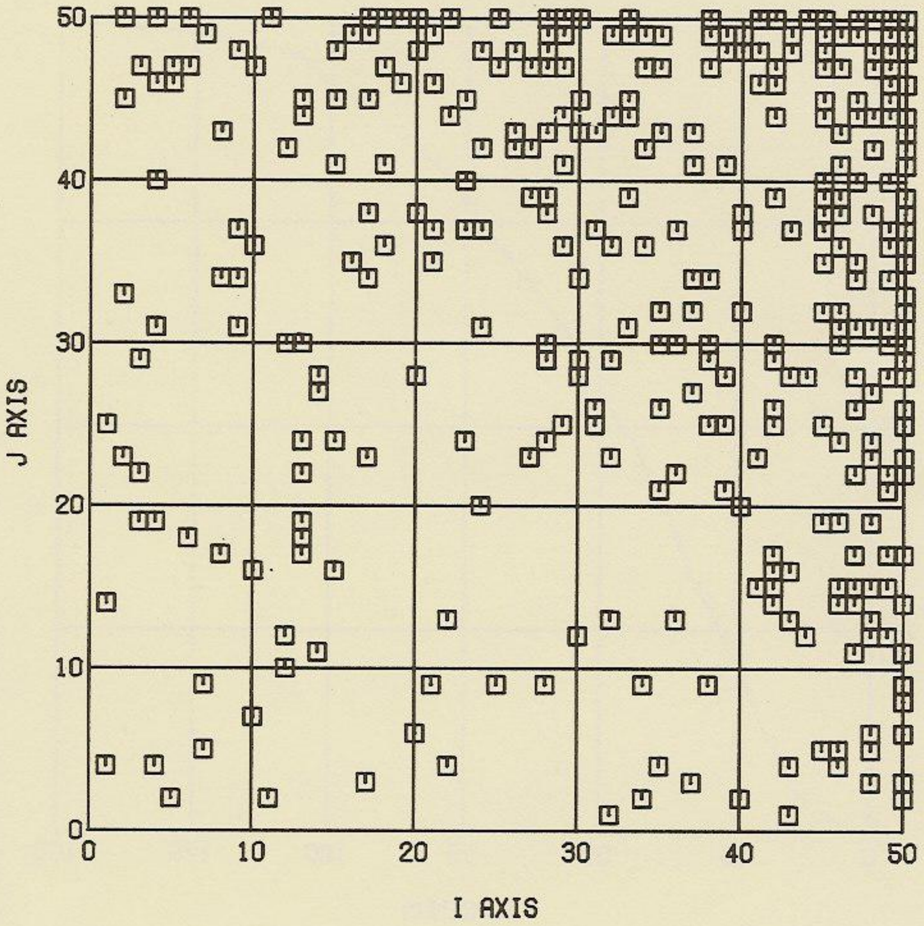
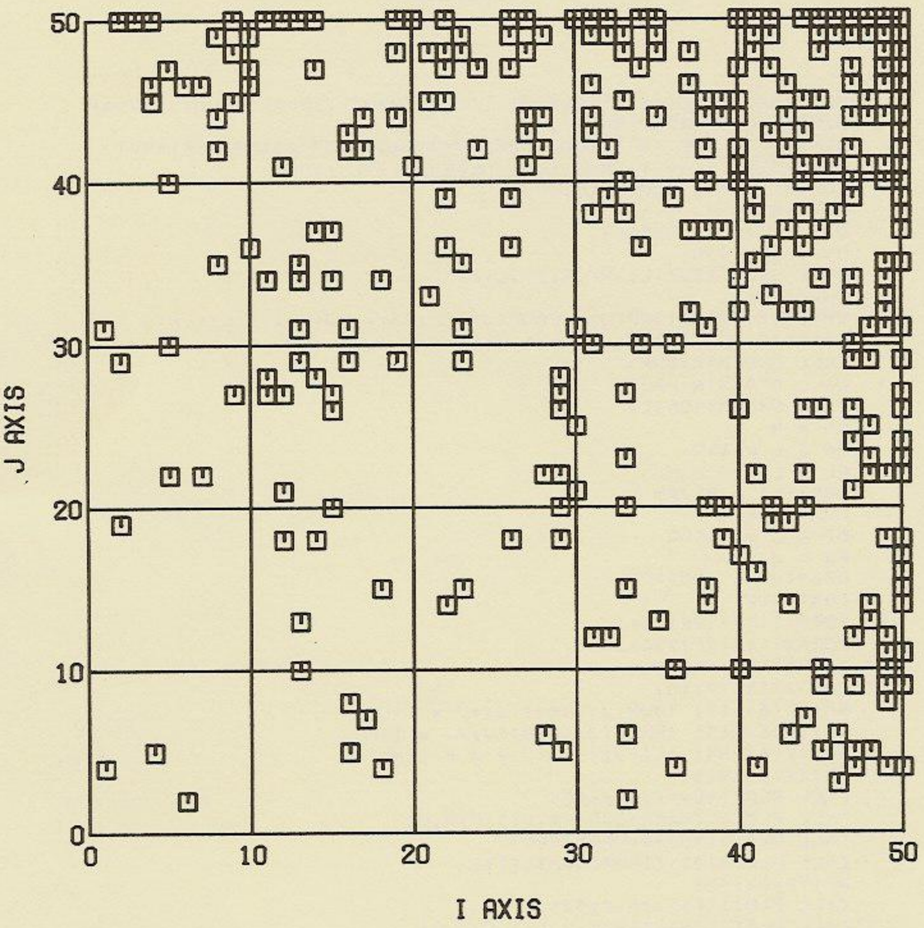


Figure 7. Location of the maxima, $\underset{\sim}{K}$, n = 400, sample 2.



APPENDIX

```
      PROGRAM RWTWO(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE8)
      DIMENSION IMAGE(1000)
      DIMENSION SUP(400),PROP(400),ORD1(400),ORD2(400),AR(400)
      DIMENSION Z(500),ORD3(500),RLI(400),RLJ(400)
      COMMON /A/ LI(400),LJ(400)
      N = 10
      DUM = RANF(7.83546)
      DO 1 L = 1,N
      CALL SHEET(SUP(L),PROP(L),L,Z)
    1 CONTINUE
      WRITE(6,102) (SUP(J), PROP(J), LI(J), LJ(J), J = 1,N)
      WRITE(6,103) (Z(J), J = 1,500)
      CALL ORDER(N,SUP)
      CALL ORDER(N,PROP)
      CALL ORDER(500,Z)
      RN = N
      DO 2 L = 1,N
      RL = L
      ORD1(L) = RL/RN
    2 CONTINUE
      DO 4 J = 1,500
      RJ = J
      ORD3(J) = .002*RJ
    4 CONTINUE
  103 FORMAT(1X,10F12.4)
 102  FORMAT(1X,2F12.4,2I6)
  113 FORMAT(1X,2F12.4)
  105 FORMAT(1X,2I10)
      WRITE(6,113) (SUP(J),ORD1(J),J = 1,N)
      WRITE(6,113) (PROP(J),ORD1(J),J = 1,N)
      WRITE(6,105) (LI(J), LJ(J), J = 1,N)
      WRITE(6,101)
      CALL PLOT1(0,2,25,4,20)
      CALL PLOT2(IMAGE,4.0,-4.0,1.0,0.0)
      CALL PLOT3(1HN,Z,ORD3,500)
      CALL PLOT4(11,11HPROBABILITY)
      WRITE(6,101)
      CALL PLOT1(0,2,25,2,50)
      CALL PLOT2(IMAGE,1.0,0.0,1.0,0.0)
      CALL PLOT3(1HP,PROP,ORD1,N)
      CALL PLOT4(11,11HPROBABILITY)
      WRITE(6,101)
  101 FORMAT(1H1)
      CALL PLOT1(0,2,25,3,33)
      CALL PLOT2(IMAGE,150.,0.0,1.0,0.0)
      CALL PLOT3(1HS,SUP,ORD1,N)
```

```
      CALL PLOT4(11,11HPROBABILITY)
      WRITE(6,101)
      DO 35 J = 1,N
      RLI(J) = LI(J)   $   RLJ(J) = LJ(J)
   35 CONTINUE
      CALL PLOT1(0,5,10,5,16)
      CALL PLOT2(IMAGE,50.,0.0,50.,0.0)
      CALL PLOT3(1HM,RLI,RLJ,N)
      CALL PLOT4(12,12HY OR T2 AXIS)
      WRITE(6,101)
      PB = 0.0
      DO 43 L = 1,N
      T = 466.25/RLI(L)
      IF(RLJ(L).GT.T)  PB = PB + 1.0
   43 CONTINUE
      PB = PB/RN
      WRITE(6,107) PB
  107 FORMAT(1X,*THE PROPORTION IN THE UPPER RIGHT CORNER IS*,F10.6)
      DO 47 J = 1,N
      WRITE(8,108)  SUP(J),PROP(J),RLI(J),RLJ(J),ORD1(J)
  108 FORMAT(1X,5F16.4)
   47 CONTINUE
      STOP
      END
```

```
      SUBROUTINE NORMAL
      COMMON S(50,50), X(50,50)
      DO 3 I = 1,50
      DO 3 J = 1,50,2
    2 A = RANF(0.0)
      B = RANF(0.0)
      U = 2.*A - 1.
      V = 2.*B - 1.
      R2 = U*U + V*V
      R = SQRT(R2)
      IF(R.GE.1.)  GO TO 2
      RLOG = ALOG(R2)
      S = SQRT(-2.*RLOG)
      X(I,J) = U*S/R
      X(I,J+1) = V*S/R
    3 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SHEET(MAX,PP,L,Z)
      COMMON S(50,50),X(50,50)
      COMMON/A/LI(400), LJ(400)
      DIMENSION Z(500)
      REAL MAX,PP
      DO 5 I = 1,50
      DO 5 J = 1,50
      S(I,J) = 0.0
    5 CONTINUE
      CALL NORMAL
      KK = -1
      IF(L.GT.1)  GO TO 15
      DO 23 J = 1,500
      Z(J) = 0.0
   23 CONTINUE
      DO 3 I = 1,10
      DO 3 J = 1,50,2
      KK = KK + 2
      Z(KK) = X(I,J)
      Z(KK+1) = X(I,J+1)
    3 CONTINUE
   15 PP = 0.0
      S(1,1) = X(1,1)
      MAX = S(1,1)
      LI(L) = 1  $  LJ(L) = 1
      DO 7 J = 2,50
      S(1,J) = S(1,J-1) + X(1,J)
      S(J,U) = S(J-1,1) + X(J,1)
      IF(S(J,1).LT.MAX)  GO TO 11
      MAX = S(J,1)
      LI(L) = J
   11 IF(S(1,J).LT.MAX)  GO TO 13
      MAX = S(1,J)
      LJ(L) = J
   13 IF(S(J,1).GT.0.0)  PP = PP + 1.0
      IF(S(1,J).GT.0.0)  PP = PP + 1.0
    7 CONTINUE
      DO 8 I = 2,50
      DO 8 J = 2,50
      S(I,J) = X(I,J) + S(I-1,J) + S(I,J-1) - S(I-1,J-1)
      IF(S(I,J).LT.MAX)  GO TO 17
      MAX = S(I,J)
      LI(L) = I   $  LJ(L) = J
   17 IF(S(I,J).GT.0.0)  PP = PP + 1.0
    8 CONTINUE
      PP = .0004*PP
      RETURN
      END
```

```
      SUBROUTINE ORDER(KK,Y)
      DIMENSION Y(500)
      M = KK
    3 M = M/2
      IF(M.EQ.0)  GO TO 8
      K = KK-M
      J = 1
    4 I = J
    5 IP = I + M
      IF(Y(I)-Y(IP)) 7,7,6
    6 T = Y(I)
      Y(I) = Y(IP)
      Y(IP) = T
      I = I - M
      IF(I-1)  7,5,5
    7 J = J+1
      IF(J-K)  4,4,3
    8 RETURN
      END
```

$$\mathrm{Var}(T_2) = \frac{1}{4}(1-\log 2)$$

$$E(T_2^2) = \frac{1}{4}(1-\log 2) + \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{4}\log 2$$

(1963)

cf Gupta, AMS 34, 829 / 792

cf Feller, II, p 527 prob #7. for CF of arcsin law.

If X has density $f_X(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$,

then $\varphi(t) \equiv Ee^{itX} = e^{it/2} J_0(t/2)$

where $J_0(s) = \sum_0^\infty \frac{(-1)^k}{k!k!}\left(\frac{1}{2}s\right)^{2k}$

$\varphi'(t) = e^{it/2}\cdot\frac{i}{2}J_0\left(\frac{t}{2}\right) + e^{it/2}J_0'\left(\frac{t}{2}\right)\cdot\frac{1}{2}$

$\varphi'(0) = \frac{i}{2}J_0(0) + \frac{1}{2}J_0'(0) = \frac{1}{2}i$

$J_0'(s) = \sum_1^\infty \frac{(-1)^k}{k!k!} 2k\left(\frac{1}{2}s\right)^{2k-1}\frac{1}{2}$

Wellner, Jon A. MR 51 #14223 14:
Monte Carlo of two-dimensional Brownian sheets.
Statistical inference and related topics (Proc. Summer Res. I. Statist. Inference for Stochastic Processes, Indiana Un. Bloomington, Ind., 1974, *Vol. 2; dedicated to Z. W. Birnbau pp.* 59–75. *Academic Press, New York*, 1975.
This paper contains a Monte Carlo study of the realizations : empirical distributions (of two functionals) of the two-dimensic Brownian sheet processes based on the approximation of latter by partial sums of normal deviates.
{For the entire collection see MR **51** #1893.}
P. K. Sen (Chapel Hill, N

$\varphi''($ … $J_0'\left(\frac{t}{2}\right)$

… $J_0''(t/2)$

$\varphi''($ … (0)

$-\left(\frac{i}{2}\right) + \frac{1}{4}\left(-\frac{1}{2}\right) = -\frac{1}{4} - \frac{1}{8} = -\frac{3}{8} = -EX^2$

$J_0''(s) = \sum_1^\infty \frac{(-1)^k}{k!k!}2k(2k-1)\left(\frac{1}{2}s\right)^{2k-2}\left(\frac{1}{4}\right)$

$J_0''(0) = \frac{(-1)^1}{1}2\cdot1\cdot\frac{1}{4} = -\frac{1}{2}$

$\mathrm{Var}\,X = \frac{3}{8} - \frac{1}{4} = \frac{1}{8} \le \frac{1}{6}$ pp. 63