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SUPPLEMENTARY MATERIAL FOR "GLOBAL RATES OF CONVERGENCE OF THE MLES OF LOG-CONCAVE AND $S-$CONCAVE DENSITIES"

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In this supplement we present additional technical arguments and proofs for [1]. Equation and theorem references made to the main document do not contain letters.

A. Appendix: Technical Lemmas and Inequalities. We begin with the proof of Proposition 4.1. It requires a result from [2], so we will state that theorem, for the reader's ease. The theorem gives bounds on bracketing numbers for classes of convex functions that are bounded and satisfy Lipschitz constraints. Let $\mathcal{C}\left([a, b],[-B, B], \Gamma\right)$ be the class of functions $f \in \mathcal{C}\left([a, b],[-B, B]\right)$ satisfying the Lipschitz constraint $|f(x)-f(y)| \leq \Gamma|x-y|$ for all $x, y \in [a, b]$.

Theorem A.1 (Theorem 3.2 of [2]). *There exist positive constants c and ϵ_0 such that for all $a<b$ and positive B, Γ, we have*

$$\log N_{[\,]}\left(\epsilon, \mathcal{C}\left([a, b],[-B, B], \Gamma\right), L_{\infty}\right) \leq c\left(\frac{B+\Gamma(b-a)}{\epsilon}\right)^{1/2}$$

for all $0<\epsilon \leq \epsilon_0\{B+\Gamma(b-a)\}$.

Proof. [2] prove this statement for metric covering numbers rather than bracketing covering numbers, but when using the supremum norm, the two are equal, if ϵ is adjusted by a factor of 2: If $f_1, \ldots, f_N$ are the centers of L_∞ balls of radius ϵ that cover a function class $\mathcal{C}$, then $[f_i-\epsilon, f_i+\epsilon]$, $i=1, \ldots, N$, are brackets of size 2ϵ that cover $\mathcal{C}$ (see e.g. page 157, the proof of Corollary 2.7.2, of [5]). □

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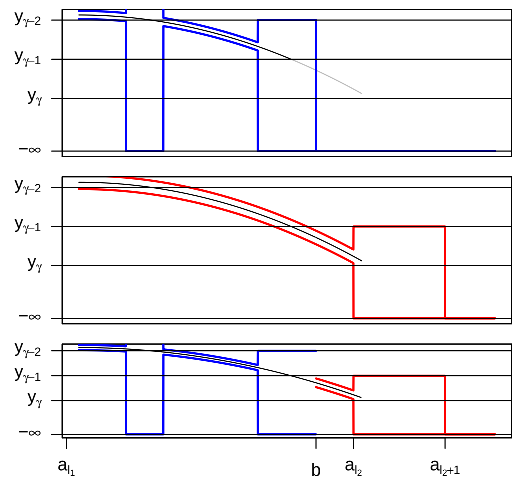
Fig A. *Theorem 4.1: Bracketing of a concave function φ (rather than $h(\varphi)$). Here $I^L_{i_\gamma,\gamma} = [a_{l_1}, a_{l_2}]$ and $I^U_{i_\gamma,\gamma} = [a_{l_1-1}, a_{l_2+1}]$, and the right boundary of the domain of φ lies between a_{l_2} and a_{l_2+1}. We focus on the right side, near a_{l_2} and a_{l_2+1}. In the top plot is a bracket on the domain $\cup_{j=1}^{\gamma-1} I^U_{i_j,j}$ (which we let have right endpoint b here) and the range $[y_{\gamma-1}, y_0]$ (below which φ is greyed out). The next plot shows an application of Proposition 4.1 to find a bracket on $I^L_{i_\gamma,\gamma}$. The final plot shows the combination of the two.*



Proof of Proposition 4.1. First, notice that the L_r bracketing numbers scale in the following fashion. For a function $f \in \mathcal{C}([b_1, b_2], [-B, B])$ we can define

$$\tilde{f}(x) := \frac{f(b_1 + (b_2 - b_1)x) - B}{B},$$

a scaled and translated version of f that satisfies $\tilde{f} \in \mathcal{C}([0,1], [-1,1])$. Thus, if $[l, u]$ is a bracket for $\mathcal{C}([b_1, b_2], [-B, B])$, then we have

$$B^r \int_0^1 \left|\tilde{u}(x) - \tilde{l}(x)\right|^r dx = \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} |u(x) - l(x)|^r \, dx.$$

Thus an $\epsilon-$size L_r bracket for $\mathcal{C}([0,1], [-1,1])$ immediately scales to be an $\epsilon(b_2 - b_1)^{1/r} B$ bracket for $\mathcal{C}([b_1, b_2], [-B, B])$. Thus for the remainder of the proof we set $b_1 = 0$, $b_2 = 1$, and $B = 1$.

We take the domain to be fixed for these classes so that we can apply Theorem 3.2 of [2] which is the building block of the proof. Now we fix

$$\mu := \exp(-2(r+1)^2(r+2)\log 2) \quad \text{and} \quad \nu := 1 - \mu. \tag{A.1}$$

(Note that μ and ν are u and v, respectively, in [2].) We will consider the intervals $[0, \mu]$, $[\mu, \nu]$, and $[\nu, 1]$ separately, and will show the bound (4.2) separately for the restriction of $\mathcal{C}([0,1],[-1,1])$ to each of these sub-intervals. This will imply (4.2). We fix $\epsilon > 0$, let $\eta = (3/17)^{1/r}\epsilon$, choose an integer A and $\delta_0, \ldots, \delta_{A+1}$ such that

$$0 = \delta_0 < \eta^r = \delta_1 < \delta_2 < \cdots < \delta_A < \mu \le \delta_{A+1}. \tag{A.2}$$

For two functions f and g on $[0,1]$, we can decompose the integral $\int_0^1 |f - g|^r d\lambda$ as

$$\int_0^1 |f-g|^r d\lambda = \int_0^\mu |f-g|^r d\lambda + \int_\mu^\nu |f-g|^r d\lambda + \int_\nu^1 |f-g|^r d\lambda. \tag{A.3}$$

The first term and last term are symmetric, so we consider just the first term, which can be bounded by

$$\int_0^\mu |f-g|^r d\lambda \le \sum_{m=0}^{A} \int_{\delta_m}^{\delta_{m+1}} |f-g|^r d\lambda, \tag{A.4}$$

since $\delta_{A+1} \ge \mu$. Now for a fixed $m \in \{1, \ldots, A\}$, we consider the problem of covering the functions in $\mathcal{C}([0,1],[-1,1])$ on the interval $[\delta_m, \delta_{m+1}]$. Defining $\tilde{f}(x) = f(\delta_m + (\delta_{m+1} - \delta_m)x)$ and $\tilde{g}(x) = g(\delta_m + (\delta_{m+1} - \delta_m)x)$, we have

$$\int_{\delta_m}^{\delta_{m+1}} |f-g|^r d\lambda = (\delta_{m+1} - \delta_m) \int_0^1 |\tilde{f} - \tilde{g}|^r d\lambda. \tag{A.5}$$

Since concavity is certainly preserved by restriction of a function, the restriction of any function f in $\mathcal{C}([0,1],[-1,1])$ to $[\delta_m, \delta_{m+1}]$ belongs to the Lipschitz class $\mathcal{C}([\delta_m, \delta_{m+1}], [-1,1], 2/\delta_m)$ (since f cannot "rise" by more than 2 over a "run" bounded by δ_m). Thus the corresponding $\tilde{f}$ belongs to $\mathcal{C}([0,1],[-1,1], 2(\delta_{m+1} - \delta_m)/\delta_m)$. We can now use Theorem A.1 to assert the existence of positive constants ϵ_0 and c that depend only on r such that for all $\alpha_m \le \epsilon_0$ there exists an α_m-bracket for $\mathcal{C}([0,1],[-1,1], 2(\delta_{m+1} - \delta_m)/\delta_m)$ in the supremum norm of cardinality smaller than

$$\exp\left(c\alpha_m^{-1/2} \left(2 + \frac{2(\delta_{m+1} - \delta_m)}{\delta_m} \right)^{1/2} \right) \le \exp\left(c \left(\frac{\delta_{m+1}}{\delta_m \alpha_m} \right)^{1/2} \right). \tag{A.6}$$

Denote the brackets by $\{[l_{m,n_m}, u_{m,n_m}] : n_m = 1, \ldots, N_m\}$ where N_m is bounded by (A.6) and $m = 1, \ldots, A$. Now, define the brackets $[l_{n_m}, u_{n_m}]$ by

$$\begin{aligned} l_{n_m}(x) &\equiv -1_{[0,\delta_1]}(x) + \textstyle\sum_{m=1}^A 1_{[\delta_m,\delta_{m+1}]}(x)\, l_{m,n_m}(x), \\ u_{n_m}(x) &\equiv 1_{[0,\delta_1]}(x) + \textstyle\sum_{m=1}^A 1_{[\delta_m,\delta_{m+1}]}(x)\, u_{m,n_m}(x) \end{aligned} \tag{A.7}$$

for the restrictions of the functions in $\mathcal{C}([0,1],[-1,1])$ to the set $[0,\mu]$, where the tuple $(n_1,\ldots,n_A)$ defining the bracket varies over all possible tuples with components $n_m \le N_m$, $m=1,\ldots,A$. The brackets were chosen in the supremum norm, so we can compute their $L_r(\lambda)$ size as $S_1^{1/r}$ where

$$S_1 = \delta_1 + \sum_{m=1}^{A} \alpha_m^r (\delta_{m+1} - \delta_m), \tag{A.8}$$

and the cardinality is $\exp(S_2)$ where

$$S_2 = c \sum_{m=1}^{A} \left(\frac{2\delta_{m+1}}{\delta_m \alpha_m} \right)^{1/2}. \tag{A.9}$$

Thus our S_1 and S_2 are identical to those in (7) in [2]. Thus, by using their choice of δ_m and α_m,

$$\delta_m = \exp\left(r \left(\frac{r+1}{r+2} \right)^{m-1} \log \eta \right),$$
$$\alpha_m = \eta \exp\left(-r \frac{(r+1)^{m-2}}{(r+2)^{m-1}} \log \eta \right),$$

their conclusion that

$$S_1 \le \frac{7}{3}\eta^r \text{ and } S_2 \le 2c \left(\frac{2}{\eta} \right)^{1/2}$$

holds.

An identical conclusion holds for the restriction of $f \in \mathcal{C}([0,1],[-1,1])$ to $[\nu,1]$. Finally, if $f \in \mathcal{C}([0,1],[-1,1])$ then its restriction to $[\mu,\nu]$ lies in $\mathcal{C}([\mu,\nu],[-B,B],2/\mu)$, for which, via Theorem A.1, for all $\eta \le \epsilon_0$, we can find a bracketing of size η in the L_r metric (which is smaller than the L_∞ metric) having cardinality smaller than

$$\exp\left(c\eta^{-1/2} \left(2 + \frac{2}{\mu} \right)^{1/2} \right) \le \exp\left(c \left(\frac{2}{\mu} \right)^{1/2} \left(\frac{2}{\eta} \right)^{1/2} \right).$$

Thus we have brackets for $\mathcal{C}([0,1],[-1,1])$ with L_r size bounded by

$$\left(\frac{7}{3}\eta^r + \frac{7}{3}\eta^r + \eta^r \right)^{1/r} = \left(\frac{17}{3} \right)^{1/r} \eta,$$

and log cardinality bounded by

$$c\left(4+\left(\frac{2}{\mu}\right)^{1/2}\right)\left(\frac{2}{\eta}\right)^{1/2}.$$

Since $\eta = (3/17)^{1/r}\epsilon$, we have shown that

$$\log N_{[\,]}(\epsilon, \mathcal{C}([0,1],[-1,1]), L_r) \le C_1\left(\frac{1}{\epsilon}\right)^{1/2}$$

for a constant C_1 and $\epsilon \le \epsilon_3 \equiv (17/3)^{1/r}\epsilon_0$.

To extend this result to all $\epsilon > 0$, we note that if $\epsilon \ge 2$, we can use the trivial bracket $[-1_{[0,1]}, 1_{[0,1]}]$. Then, letting $C_2 = \frac{(1/\epsilon_3)^{1/2}}{1/2^{1/2}}$, for $\epsilon_3 \le \epsilon \le 2$ we have

$$C_2 \cdot C_1 \epsilon^{-1/2} \ge C_1 \epsilon_3^{-1/2} \ge \log N_{[\,]}(\epsilon, \mathcal{C}([0,1],[-1,1]), L_r),$$

since bracketing numbers are non-increasing. Thus, taking $C \equiv C_2 \cdot C_1$, we have shown (4.2) holds for all $\epsilon > 0$ with $[b_1, b_2] = [0,1]$ and $B = 1$. By the scaling argument at the beginning of the proof we are now done. $\square$

For $\delta > 0$ and $\mathcal{P}_h$ consisting of all h-concave densities on $\mathbb{R}$ as in (4.1), let

$$\mathcal{P}_h(\delta) \equiv \{p \in \mathcal{P}_h : \; H(p, p_0) < \delta\},$$

$$\overline{\mathcal{P}}_h(\delta) \equiv \{(p+p_0)/2 : \; p \in \mathcal{P}_h, H((p+p_0)/2, p_0) < \delta\},$$

and let $\mathcal{P}_{M,h}$ be as defined in (4.3).

LEMMA A.1. *Let $\delta > 0$ and $0 < \epsilon \le \delta$. With the definitions in the previous display*

$$N_{[\,]}(\epsilon, \overline{\mathcal{P}}_h(\delta), H) \lesssim N_{[\,]}(\epsilon, \mathcal{P}_h(4\delta), H) \tag{A.10}$$

$$< N_{[\,]}(\epsilon, \mathcal{P}_{M,h}, H). \tag{A.11}$$

PROOF. We will follow the notation in [4] (see e.g. chapter 4) and set $\overline{p} = (p+p_0)/2$ for any function p. Then if $\overline{p}_1 \in \overline{\mathcal{P}}_h(\delta)$, by (4.6) on page 48 of [4], we have $H(p_1, p_0) < 4H(\overline{p}_1, p_0) < 4\delta$, so that $p_1 \in \mathcal{P}_h(4\delta)$. Then given $\epsilon-$brackets $[l_\alpha, u_\alpha]$, of $\mathcal{P}_h(4\delta)$, with $1 \le \alpha \le N_{[\,]}(\epsilon, \mathcal{P}_h(4\delta), H)$, we can construct brackets of $\overline{\mathcal{P}}_h(\delta)$ since for any $p_1 \in \mathcal{P}_h(4\delta)$ which is bracketed by $[l_\alpha, u_\alpha]$ for some α, $\overline{p_1}$ is bracketed by $[\bar{l}_\alpha, \overline{u}_\alpha]$, so that $[\bar{l}_\alpha, \overline{u}_\alpha]$ form a collection of brackets for $\overline{\mathcal{P}}_h(\delta)$ with size bounded by

$$H(\bar{l}_\alpha, \overline{u}_\alpha) \le \frac{1}{\sqrt{2}} H(l_\alpha, u_\alpha) < \frac{1}{\sqrt{2}}\epsilon,$$

where we used (4.5) on page 48 of [4]. Thus we have a collection of brackets of Hellinger size $\epsilon/\sqrt{2} < \epsilon$ with cardinality bounded by $N_{[\,]}(\epsilon, \mathcal{P}_h(4\delta), H)$ and (A.10) holds.

Next we show (A.11), which will follow from showing $\mathcal{P}_h(4\delta) \subset \mathcal{P}_{M,h}$. Now if $0 < M^{-1} < \inf_{x\in[-1,1]} p_0(x)$ then for any p that has its mode in $[-1,1]$ and satisfies

(A.12)
$$\sup_{x\in[-1,1]} |p(x) - p_0(x)| \le \min\left(\inf_{x\in[-1,1]} p_0(x) - M^{-1}, M - \sup_{x\in[-1,1]} p_0(x) \right),$$

we can conclude that $p \in \mathcal{P}_{M,h}$.

The proof of Lemma 3.14 of [3] shows that for any sequence of h-concave densities p_i,

$$H(p_i, p_0) \to 0 \quad \text{implies} \quad \sup_{x\in[-1,1]} |p_i(x) - p_0(x)| \to 0. \tag{A.13}$$

This says that the topology defined by the Hellinger metric has more open sets than that defined by the supremum distance on $[-1,1]$, which implies that open supremum balls are nested within open Hellinger balls, i.e. for $\epsilon > 0$

$$B_\epsilon(p_0, \sup_{[-1,1]}) \subseteq B_{4\delta}(p_0, H) \tag{A.14}$$

for some $\delta > 0$, where $B_\epsilon(p_0, d)$ denotes an open ball about p_0 of size ϵ in the metric d.

Now, if p is uniformly within ϵ of p_0 on $[-1,1]$, then for ϵ small enough we know that the mode of p is in $[-1,1]$. Thus for $0 < M^{-1} < \inf_{x\in[-1,1]} p_0(x)$ and δ small enough, any $p \in \mathcal{P}_h(4\delta)$ is also in $\mathcal{P}_{M,h}$ as desired, and so (A.11) has been shown. □

LEMMA A.2. *For a concave-function transformation h that satisfies Assumption T.1, we can have that h^{-1} is nondecreasing and as $f \searrow 0$,*

$$h^{-1}(f) = o(f^{-1/\alpha}). \tag{A.15}$$

In particular, for $f \in (0, L]$, $h^{-1}(f) \le M_L f^{-1/\alpha}$.

PROOF. Let $\operatorname{ran} h = h(\operatorname{dom} h)$. For two increasing functions $h \le g$ defined on $(-\infty, \infty)$ taking values in $[-\infty, \infty]$, where $\operatorname{ran} h$ and $\operatorname{ran} g$ are both intervals, we will show that $g^{-1}(f) \le h^{-1}(f)$ for any $f \in \operatorname{ran} h \cap \operatorname{ran} g$. By definition, for such f, we can find a $z \in (-\infty, \infty)$ such that $f = g(z)$.

That is, $g(z) = h(h^{-1})(f) \le g(h^{-1}(f))$ since $h \le g$. Applying g^{-1}, we see $z = g^{-1}(f) \le h^{-1}(f)$, as desired.

Then (A.15) follows by letting $g(y) = \delta(-y)^{-\alpha}$, which has $g^{-1}(f) = -(\frac{1}{\delta} f)^{-1/\alpha}$. The statement that $h^{-1}(f) \le M_L f^{-1/\alpha}$ follows since on neighborhoods away from 0, h^{-1} is bounded above and $f \mapsto f^{-1/\alpha}$ is bounded below.

To see that h^{-1} is nondecreasing, we differentiate to see $(h^{-1})'(f) = 1/h'(h^{-1}(f))$. Since $h' \ge 0$ so is $(h^{-1})'$. □

Proposition A.1. *Let h be a concave-function transformation and $f = h \circ \varphi$ for $\varphi \in \mathcal{C}$ and let $F(x) = \int_{-\infty}^x f(y)\,dy$. Then for $x_0 < x_1 < x$ or $x < x_1 < x_0$, all such that $-\infty < \varphi(x) < \varphi(x_1) < \varphi(x_0) < \infty$, we have*

$$f(x) \le h\left(\varphi(x_0) - h(\varphi(x_1))\frac{\varphi(x_0) - \varphi(x_1)}{F(x) - F(x_0)}(x - x_0)\right). \tag{A.16}$$

Proof. Take $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. Then

$$\begin{aligned} F(x_2) - F(x_1) &= \int_{x_1}^{x_2} f(x)\,dx = \int_{x_1}^{x_2} h(\varphi(x))\,dx \\ &= \int_{x_1}^{x_2} h\left(\varphi\left(\frac{x_2 - x}{x_2 - x_1}x_1 + \frac{x - x_1}{x_2 - x_1}x_2\right)\right)\,dx, \end{aligned}$$

and since h is nondecreasing and φ is concave, the above is not smaller than

$$\int_{x_1}^{x_2} h\left(\frac{x_2 - x}{x_2 - x_1}\varphi(x_1) + \frac{x - x_1}{x_2 - x_1}\varphi(x_2)\right)\,dx,$$

which, by the change of variables $u = (x - x_1)/(x_2 - x_1)$, can be written as

$$\int_0^1 h\left((1 - u)\varphi(x_1) + u\varphi(x_2)\right)(x_2 - x_1)\,du. \tag{A.17}$$

Now we let $x_1 = x_0$ and $x_2 = x$ with $x_0 < x_1 < x$ as in the statement. Since x_0 and x_1 are in $\operatorname{dom}\varphi$,

$$C \equiv \int_0^1 h((1 - u)\varphi(x_0) + u\varphi(x_1))\,du \tag{A.18}$$

satisfies

$$0 < h(\varphi(x_1)) \le C \le h(\varphi(x_0)). \tag{A.19}$$

Now, let $\eta = (\varphi(x_0) - \varphi(x_1))/(\varphi(x_0) - \varphi(x))$, so that $\eta \in (0,1)$ by the assumption of the proposition. Then

$$\begin{aligned}
&\int_0^1 h((1-u)\varphi(x_0) + u\varphi(x))\,du \\
&= \left(\int_0^\eta + \int_\eta^1\right) h((1-u)\varphi(x_0) + u\varphi(x))\,du \\
&\geq \int_0^\eta h((1-u)\varphi(x_0) + u\varphi(x))\,du.
\end{aligned}$$

Then by the substitution $v = u/\eta$, this is equal to

$$\int_0^1 h\left((1-\eta v)\varphi(x_0) + \eta v\varphi(x)\right)\eta\,dv. \tag{A.20}$$

which is

$$\int_0^1 h\left((1-v)\varphi(x_0) + v\varphi(x_1)\right)\frac{\varphi(x_0) - \varphi(x_1)}{\varphi(x_0) - \varphi(x)}\,dv, \tag{A.21}$$

by the construction of η, i.e. because

$$\begin{aligned}
(1-\eta v)\varphi(x_0) + \eta v\varphi(x) &= \left(1 - \frac{\varphi(x_0) - \varphi(x_1)}{\varphi(x_0) - \varphi(x)}v\right)\varphi(x_0) + \frac{\varphi(x_0) - \varphi(x_1)}{\varphi(x_0) - \varphi(x)}v\varphi(x) \\
&= v\varphi(x_0) + \frac{\varphi(x_0) - \varphi(x_1)}{\varphi(x_0) - \varphi(x)}v(\varphi(x) - \varphi(x_0)) \\
&= v\varphi(x_0) - v(\varphi(x_0) - \varphi(x_1)) \\
&= (1-v)\varphi(x_0) + v\varphi(x_1).
\end{aligned}$$

And, by definition of C, (A.21) equals $C(\varphi(x_0) - \varphi(x_1))/(\varphi(x_0) - \varphi(x))$. This gives, by applying (A.17), that

$$\begin{aligned}
F(x) - F(x_0) &\geq (x - x_0)\int_0^1 h((1-u)\varphi(x_0) + u\varphi(x))\,du \\
&\geq (x - x_0)C\frac{\varphi(x_0) - \varphi(x_1)}{\varphi(x_0) - \varphi(x)}. \qquad \text{(A.22)}
\end{aligned}$$

Now we rearrange the above display to get an inequality for $\varphi(x)$. From (A.22), we have

$$\varphi(x) \leq \varphi(x_0) - C\frac{\varphi(x_0) - \varphi(x_1)}{F(x) - F(x_0)}(x - x_0),$$

and, since h is nondecreasing,

$$h(\varphi(x)) \leq h\left(\varphi(x_0) - C\frac{\varphi(x_0)-\varphi(x_1)}{F(x)-F(x_0)}(x-x_0)\right)$$
$$\leq h\left(\varphi(x_0) - h(\varphi(x_1))\frac{\varphi(x_0)-\varphi(x_1)}{F(x)-F(x_0)}(x-x_0)\right),$$

by (A.19). This proves the claim for $x_0 < x_1 < x$. The proof for $x < x_1 < x_0$ is similar. $\square$

LEMMA A.3. *If $g \equiv h^{1/2}$ is a concave-function transformation satisfying $g'(y) = o(|y|^{-(\alpha_g+1)})$ then $g(y) = o(|y|^{-\alpha_g})$, $h(y) = o(|y|^{-2\alpha_g})$, and $h'(y) = o(|y|^{-(2\alpha_g+1)})$ as $y \to -\infty$.*

PROOF. Since for any $\delta > 0$ we can find $N > 0$ where for $y < -N$, $g(x) = \int_{-\infty}^{x} g'(y)dy \leq \delta \int_{-\infty}^{x}(-y)^{-(\alpha_g+1)}$, we conclude that $g(y) = o(|y|^{-\alpha_g})$. It follows additionally that $h(y) = o(|y|^{-2\alpha_g})$. Thus for $\delta > 0$ there exists N such that for $y < -N$, $h^{-1/2}(y) \geq \delta^{-1/2}|y|^{\alpha}$, and so we have that

$$\delta|y|^{-(\alpha_g+1)} \geq h^{-1/2}(y)h'(y) \geq \delta^{-1/2}|y|^{\alpha_g}h'(y)$$

since $2g'(y) = h^{-1/2}(y)h'(y)$, so that $\delta^{3/2}|y|^{-(2\alpha_g+1)} \geq h'(y)$, as desired. $\square$

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