On Longest Increasing Subsequences and Random Young Tableaux: Experimental Results and Recent Theorems

Jon A. Wellner

University of Washington

April 9, 2002

Abstract

Ulam (1961) apparently first posed the following question: what is the average (or distribution of) the length L_n of the longest increasing subsequence of a random permutation of the first n integers? Experimental (Monte-Carlo) evidence has played an important role in the study of L_n begining with Ulam (1961), and continuing with Baer and Brock (1969), and Odlyzko and Rains (2000). We present experimental evidence concerning the distribution of the length L_n of the longest increasing increasing subsequence of a random permutation of length n. In particular, the experimental data confirm the known result $E(L_n) \sim 2\sqrt{n}$ and strongly suggest that $Var(L_n) \sim cn^{1/3}$ for a constant $c \approx .818\ldots$. This supports and complements the recent results of Baik, Deift, and Johansson (1999), who apparently knew of the monte-carlo results of Odlyzko and Rains (2000). In the last section we also combine our experimental results with those of Odlyzko and Rains (2000).

AMS 2000 *subject classifications.* Primary: 60G15, 60G99; secondary 60E05.

Key words and phrases. limiting distributions, longest increasing subsequence, mean, monte-carlo method, random permutation, random Young Tableaux, random matrices, Ulam's metric, variance.

1 Introduction.

For a recent survey of this problem with connections to "patience sorting", see ALDOUS AND DIACONIS (1999).

2 History, part 1.

Let $\Pi_n = (\Pi_{n1}, \ldots, \Pi_{nn})$ be a random permutation of the first n integers $\{1, \ldots, n\}$. Thus

$$P(\Pi_n = \pi_n) = \frac{1}{n!} \qquad \text{for all permutations } \ \pi_n \ \text{ of } \ \{1, \ldots, n\}\,.$$

Let $L_n = L_n(\Pi_n)$ be the length of the longest increasing subsequence in Π_n. For example, if $\Pi \equiv \Pi_9 = (3, 2, 1, 4, 5, 9, 6, 7, 8)$, then $L_9 = 6$ for this outcome since $(\Pi_3, \Pi_4, \Pi_5, \Pi_7, \Pi_8, \Pi_9) = (1, 4, 5, 6, 7, 8)$ is an increasing subsequence in Π of length 6. Note that this subsequence is not unique: in the present example $(\Pi_2, \Pi_4, \Pi_5, \Pi_7, \Pi_8, \Pi_9) = (2, 4, 5, 6, 7, 8)$ and $(\Pi_1, \Pi_4, \Pi_5, \Pi_7, \Pi_8, \Pi_9) = (3, 4, 5, 6, 7, 8)$ are also increasing subsequences in Π of length 6.

The length of the longest increasing subsequence is also the basis of a metric between permutations: if $\pi = \pi_n$ and $\sigma = \sigma_n$ are two permutations of $\{1, \ldots, n\}$, then ULAM (1972) defined $U(\pi, \sigma) = n - L_n(\sigma \circ \pi^{-1})$; see also ULAM (1981), DIACONIS (1982) pages 118-119, and CRITCHLOW (1985).

ULAM (1961) posed the question:

Question 1: How fast does $E(L_n)$ grow with n?

ULAM (1961) gave a preliminary monte-carlo analysis of this question, but the first steps towards a definitive answer were first found by BAER AND BROCK (1968). and by HAMMERSLEY (1972). BAER AND BROCK (1968) tabulated the distributions of L_n and of $\max\{L_n, K_n\}$, where K_n is the length of the longest *decreasing* subsequence, for n between 1 and 36, and then computed estimates of $E(L_n)$ for n up to $10,000$; see their Figure 1, and note the plotted line $2\sqrt{n}$. The exact calculations of BAER AND BROCK (1968) were extended to $n \le 75$ by MACKAY (1976), and to $n \le 120$ by ODLYZKO AND RAINS (2000).

HAMMERSLEY (1972) used Kingman's subadditive ergodic theorem and a clever embedding of the problem in a bivariate Poisson process to show that

$$\frac{L_n}{\sqrt{n}} \to_p c \qquad \text{and} \qquad \frac{E(L_n)}{\sqrt{n}} \to c$$

for a constant $c \in (\pi/2, e)$, but he was unable to prove that the constant $c = 2$ as suggested by the calculations of BAER AND BROCK (1968).

3 History, part 2: Young tableaux.

SCHENSTED (1961) showed that there is a one-to-one correspondence between a random permuation Π_n and a pair of *Young tableaux* of the same shape

For our example, $\Pi \equiv \Pi_9 = (3, 2, 1, 4, 5, 9, 6, 7, 8)$, one of the corresponding Young tableau, Λ_9, is given (in the "English style") by

```
1 4 5 6 7 8
2 9
3
```

or, rotating the axes by 90° (in the "French style"),

```
8
7
6
5
4 9
1 2 3
```

The *hook lengths* are the numbers of elements in the Young tableau above and to the right of each cell in this version of the tableau: thus the hook lengths $\{h_{ij}\}$ for the above tableau are

```
  1           1           1
  2           2           2
  3           3           3
  4           4           5 1
  6 1         5           6 2
  8 3 1       9 3 2 1     7 3

  3456        6480        7560   hook product
.03038      .00864      .00634   probability
```

The corresponding *hook product* is

$$\prod_{(i,j)\in\Lambda_n} h_{ij} = 8\cdot 6\cdot 4\cdot 3\cdot 2\cdot 1\cdot 3\cdot 1\cdot 1 = 3456\,.$$

Frame, Robinson, and Thrall (1954) showed that the number of (standard) tableaux of a given shape λ_n is

$$\frac{n!}{\prod_{(i,j)\in\Lambda_n} h_{ij}} \equiv \frac{n!}{h(\lambda_n)}\,.$$

As a consequence of this together with the Schensted correspondence, for any given shape λ_n,

$$P(\Lambda_n = \lambda_n) = \frac{n!}{h^2(\lambda_n)} = \frac{1}{n!}\,\frac{n!}{h(\lambda_n)}\,\frac{n!}{h(\lambda_n)}\,.$$

For our example with $n = 9$, $n! = 362880$, the shape corresponding to our given permutation π has probability

$$P(\Lambda_9 = \lambda) = \frac{362880}{(3456)^2} = .03038\,.$$

Summing over all shapes λ with a fixed height of the first column, yields the distribution of L_n:

$$P(L_n = k) = \sum_{\lambda(0)=k} P(\Lambda_n = \lambda) = \sum_{\lambda(0)=k} \frac{n!}{h^2(\lambda)}\,.$$

For our example,

$$P(L_9 = 6) = \sum_{i=1}^{3} P(\Lambda_9 = \lambda_i) = \frac{16465}{362880} = .04537\,,$$

in agreement with the tabled probability in Baer and Brock (1968), Table 1a, page 390.

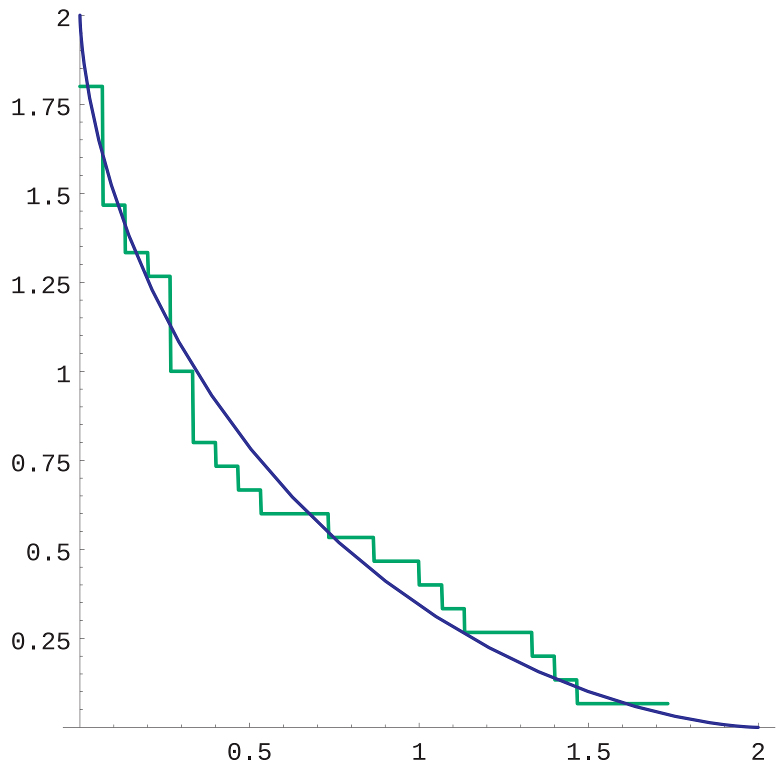

Figure 1: Scaled Young tableau with limiting shape.

The corresponding *cumulative tableau* is given by

```
    3 . . .
  9 9
  2 2
8 8 8 . . .
7 7 7
6 6 6
5 5 5 . . .
4 4 4 . . .
1 1 1 . . .
```

Because the tableaux Λ_n have total area n, it is useful to rescale the picture to have total area 1. Thus if $\Lambda_n(t)$ is the (random) function giving the upper boundary of the tableaux in the plane (with total area n), set

$$f_n(t) = n^{-1/2}\Lambda_n(tn^{1/2}), \qquad t \in [0, \infty)\,.$$

Figure 1 shows an example of a scaled random Young tableaux for $n = 625$ together with the limit function of Logan and Shepp (1977) Note that the area under f_n is 1 for every n so that f_n can be viewed as a random density function on $[0, \infty)$.

Similarly, corresponding to the cumulative tableau displayed above, set

$$F_n(t) = \int_0^t f_n(s)ds\,.$$

Thus F_n is a (random) distribution function on $[0,\infty)$ with $F_n(0) = 0$ and $F_n(\infty) = 1$. Logan and Shepp (1977) found the "shape" λ which maximizes

$$P(\Lambda_n = \lambda) = \frac{n!}{h^2(\lambda)}\,;$$

equivalently, minimize

$$\begin{aligned}\log h(\lambda) &= \sum_{(i,j)\in\Lambda_n} \log h_{ij} \\ &= \sum_{S\in\Lambda_n} \log\{\lambda(x_S) - y_S + \lambda^{-1}(y_S) - x_S\}\,.\end{aligned}$$

After rescaling by $\sqrt{n}$, and passing to the limit this becomes equivalent to finding the "shape" (a non-increasing function from $[0,\infty)$ to $[0,\infty)$) f which minimizes

$$H(f) \equiv \int_0^\infty \int_0^{f(x)} \log\{f(x) - y + f^{-1}(y) - x\}dydx\,. \tag{1}$$

subject to

$$\int_0^\infty f(x)dx = 1\,. \tag{2}$$

Theorem 1. (Logan and Shepp, 1977). For $t \in (0,2)$,

$$f_n(t) \to_p f_0(t)$$

(with respect to a complicated metric d) where f_0, given by

$$f_0(t) = \frac{2}{\pi}(\sin\theta - \theta\cos\theta)\,, \tag{3}$$

$$t = \frac{2}{\pi}(\sin\theta - \theta\cos\theta) + 2\cos\theta\,, \tag{4}$$

$0 \le \theta \le \pi$, minimizes H in (1) subject to (2); i.e.

$$H(f_0) \le H(f) \qquad \text{for all } f \text{ with } \int_0^\infty f(x)dx = 1\,.$$

Logan and Shepp (1977) used this to show that the constant c in Hammersley's theorem satisfies $c \ge 2$. Now define $g_n(t)$ by rotating the coordinate system for f_n by 45° and scaling both new axes by $\sqrt{2}$.

Theorem 2. (Vershik and Kerov, 1977). For $t \in (0,2)$,

$$g_n(t) \to_p g_0(t),$$

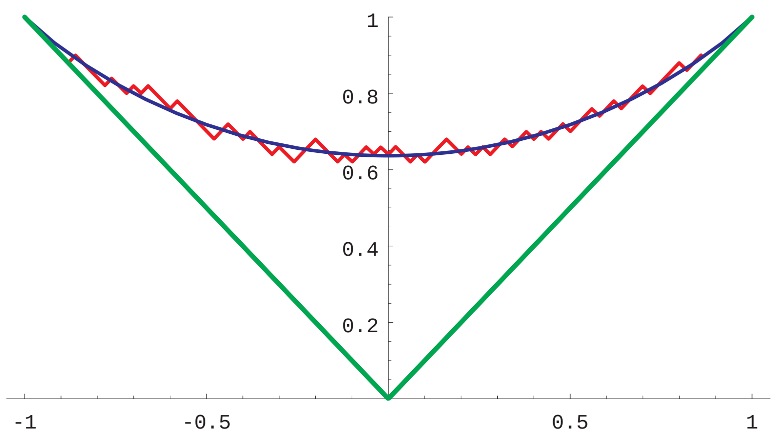

Figure 2: Limit function with coordinate system rotated by 45°.

where g_0 is given by

$$g_0(t) = \begin{cases} \frac{2}{\pi}(t \arcsin(t) + \sqrt{1-t^2}), & |t| \le 1 \\ |t|, & |t| \ge 1\,. \end{cases}$$

Note that g_0 is just the function f_0 in the new coordinate system with both axes scaled by $\sqrt{2}$. Vershik and Kerov (1977) succeeded in using this to show that $c \le 2$ and in fact that $c = 2$. Figure 2 gives a plot of the limiting shape g_0 in the rotated coordinate system.

By now there are several other proofs that $c = 2$: Aldous and Diaconis (1995) prove it again via a limit theorem for an interacting particle system process, the *Hammersley process*. Seppäläinen (1996) attacks the problem by way of "stick breaking" and a related "inviscid Burgers equation.

4 History, part 3: how big is $Var(L_n)$?

The next questions about L_n are, quite naturally:

Question 2: How fast does $Var(L_n)$ grow with n?

Question 3: Does $n^{\alpha}(L_n/\sqrt{n} - 2)$ converge in distribution for some α?

For some time there was little progress on the theoretical front, but partial results were obtained by Pilpel (1990), Talagrand (1995), Bollobás and Janson (1997), and Kim (1996). Meanwhile Odlyzko and Rains (2000), in an effort apparently dating back to about 1994, carried out an extensive monte-carlo study of $L_n = L_n^{(1)}$ and $L_n^{(2)}$, the height of the second row of the Young tableaux, with n ranging up to 10^{10}. [Note: I did not see the paper Odlyzko and Rains (2000) until April 2002.]

Meanwhile, because of some analogies with the Grenander estimator of a monotone density (Grenander (1956), Groeneboom (1985), Groeneboom (1989)), I made the following conjecture in 1996 or 1997, and started the monte-carlo study reported here, finally getting up to $n = 3 \times 10^7$ with a C program running on a cluster of Pentium machines under Linux.

Conjecture 1.

$$n^{1/3}(\frac{L_n}{\sqrt{n}} - 2) \to_d Z$$

for some (tight) random variable Z with $EZ < 0$ and $Var(Z) < \infty$.
Assuming Conjecture 1 is *true*, then it would follow that

$$-E\left\{L_n - 2\sqrt{n}\right\} \simeq -n^{1/6}E(Z)$$

and

$$Var\left(L_n - 2\sqrt{n}\right) \simeq n^{1/3}Var(Z)$$

and hence both

$$\log(-E\left\{L_n - 2\sqrt{n}\right\}) \simeq \log(-EZ) + \frac{1}{6}\log n$$

and

$$\log\left(L_n - 2\sqrt{n}\right) \simeq \log(Var(Z)) + \frac{1}{3}\log n.$$

Thus, *if* Conjecture 1 is correct, plots of means and variances of $L_n - 2\sqrt{n}$ respectively would yield slopes of 1/6 and 1/3 in $\log - \log$ plots, and the intercepts in these plots will yield $\log(-E(Z))$ and $\log(Var(Z))$.

5 The Monte-Carlo Data and Plots.

To obtain experimental evidence in favor of Conjecture 1, I carried out a Monte-Carlo study. The C-program for calculating random permutations Π_n and their lengths L_n was obtained from David Eppstein, University of California at Irvine, and put into operation by Greg Warnes.

Table 1. Summary Data from the Monte-Carlo Experiment;
Means and Variances of L_n;
10^4 monte-carlo replications for each n

n	mean	variance	n	mean	variance
10	1.9846	.7940	30000	9.3761	24.1153
20	2.4074	1.4275	40000	9.7963	27.0503
30	2.5638	1.4651	50000	10.2146	28.3498
40	2.7075	1.7717	60000	10.5591	30.8597
50	2.8080	1.9632	70000	10.8896	32.7036
60	2.9246	2.1264	80000	11.1427	33.3355
70	3.0463	2.2318	90000	11.1930	35.0279
80	3.1203	2.4505	100000	11.4946	36.8715
90	3.1808	2.6093	200000	13.0704	46.0337
100	3.2948	2.6673	300000	14.1246	51.8134
200	3.7251	3.6400	400000	14.7358	60.0396
300	4.0184	4.2844	500000	15.3908	63.0627
400	4.2123	4.8889	600000	15.7583	67.7113
500	4.4615	5.3338	700000	16.2124	71.1916
600	4.5971	5.6174	800000	16.5686	73.1944
700	4.7434	6.0837	900000	16.8825	77.5627
800	4.8923	6.4946	1000000	17.2997	81.6902
900	4.9798	6.8566	2000000	19.3319	100.5800
1000	5.0660	7.0709	3000000	20.7775	116.3620
2000	5.7218	9.0217	4000000	21.7820	126.7600
3000	6.2123	10.5477	5000000	22.4748	141.4180
4000	6.5214	11.8464	6000000	23.5006	146.7880
5000	6.8619	12.6437	7000000	24.1026	153.6630
6000	7.0178	13.5061	8000000	24.4065	161.0180
7000	7.2737	14.6082	9000000	25.2880	176.0870
8000	7.4252	15.0599	10000000	25.6300	177.1350
9000	7.6560	15.4328	20000000	29.0000	215.5790
10000	7.7355	15.6385	30000000	30.7069	256.0870
20000	8.7173	21.0034			

All the data generated in the experiments are available at

`http://www.stat.washington.edu/jaw/RESEARCH/TABLES/table.list.html` .

Figures 3 and 4 display the empirical means and variances of $(2\sqrt{n} - L_n)$ plotted versus n. The straight lines were obtained by varying the intercept and keeping the slopes fixed at 1/6 and 1/3 respectively. Figure 5 displays the empirical means of $L_n/\sqrt{n}$.

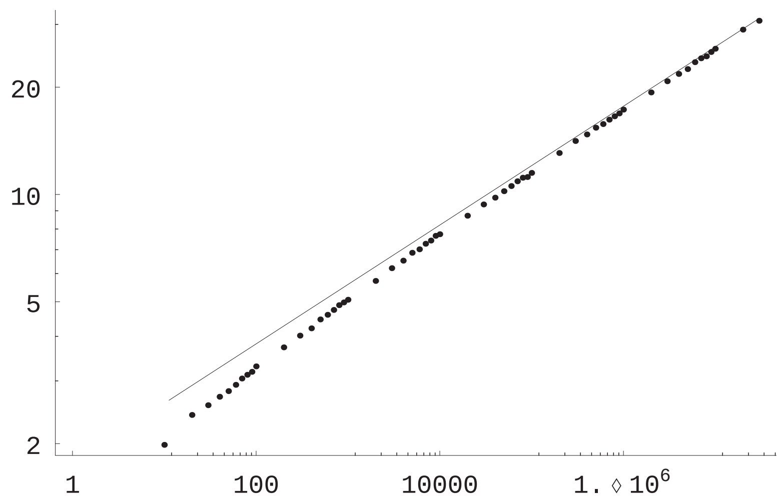

Figure 3: Empirical means of $-(L_n - 2\sqrt{n})$ and line with slope 1/6.

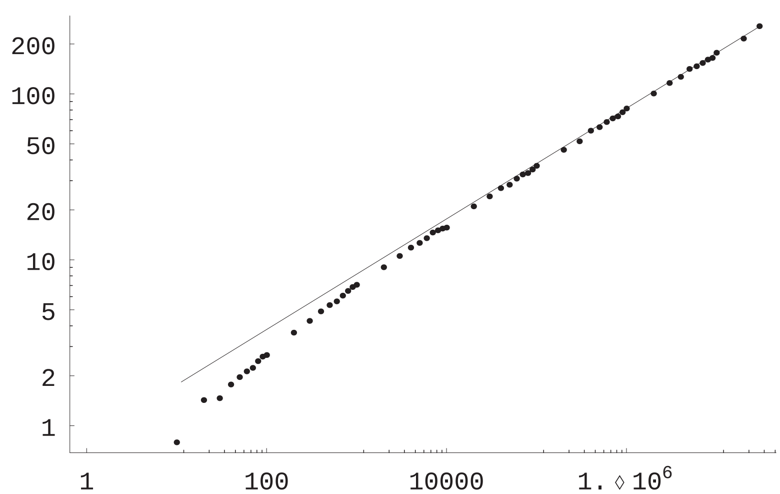

Figure 4: Empirical variances of $-(L_n - 2\sqrt{n})$ and line with slope 1/3.

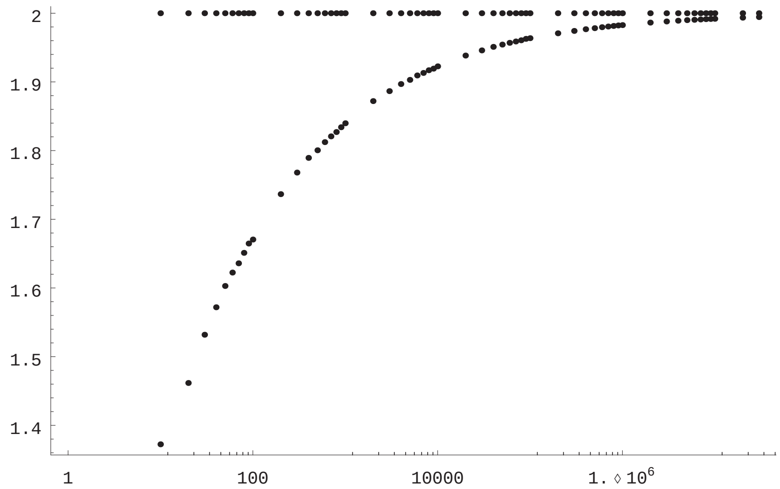

Figure 5: Empirical means of $L_n/\sqrt{n}$.

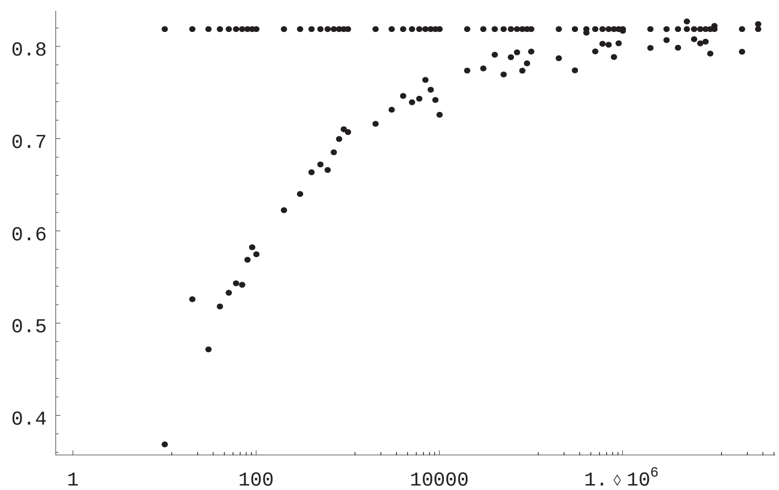

Figure 6: Empirical variances of $n^{1/3}(L_n/\sqrt{n}-2)$.

Table 2. Estimated constants (Wellner, Odlyzko and Rains); and theoretical results from Baik, Deift, Johansson (1999) and Tracy and Widom (1994)

n	Estimated $-E(Z)$	Estimated $Var(Z)$
5×10^5	1.727	.794
1×10^7	1.746	.822
3×10^7	1.742	.824
fitted lines	1.768	.818
estimates (OR)	1.758	.819
theory (BDJ)	1.7711	.8132

6 History, part 4: Conjecture 1 is true!

In fact Conjecture 1 is true, as has been proved by Baik, Deift, and Johansson (1999).

Theorem 3. (Baik, Deift, and Johansson, 1999).

$$n^{1/3}(L_n/\sqrt{n} - 2) \to_d Z$$

where

$$P(Z \le z) = \exp\left(- \int_z^\infty (x - z) u^2(x)\, dx \right) \equiv F(z)$$

and the function u satisfies the Painlevé II equation

$$u''(x) = xu(x) + 2u^3(x)$$

subject to the boundary condition

$$u(x) \sim -Ai(x) \qquad \text{as } x \to \infty\,.$$

By earlier results of Tracy and Widom (1994), the solution u exists, is unique,

$$u(x) = -Ai(x) + O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}} \right) \qquad x \to \infty\,,$$

and

$$u(x) = -\sqrt{\frac{|x|}{2}}\left(1 + O(x^{-2})\right) \qquad \text{as } x \to -\infty\,.$$

By numerical calculations, Tracy and Widom (1994) and Baik, Deift, and Johansson (1999) find that for $Z \sim F$, $E(Z) = -1.7711...$ and $Var(Z) = 0.8132...$. For discussions of further related results, see Aldous and Diaconis (1999), Deift (2000), and the lecture notes of Tracy (2001).

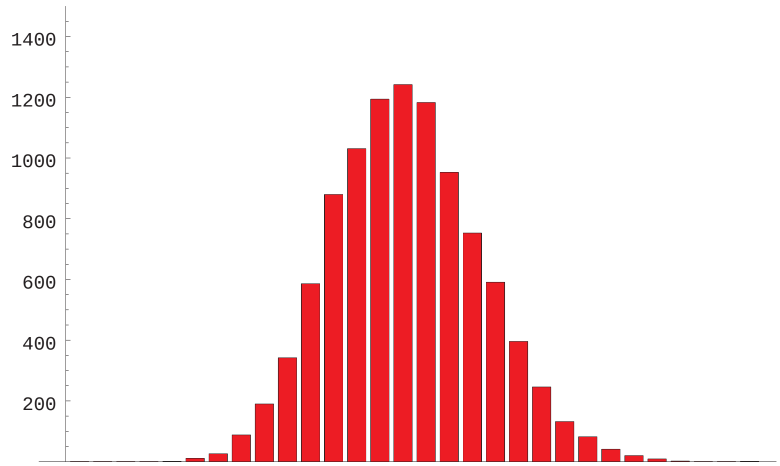

Figure 7: Histogram of 10^4 simulated values of $n^{1/3}(L_n/\sqrt{n}-2)$ with $n = 3 \times 10^7$.

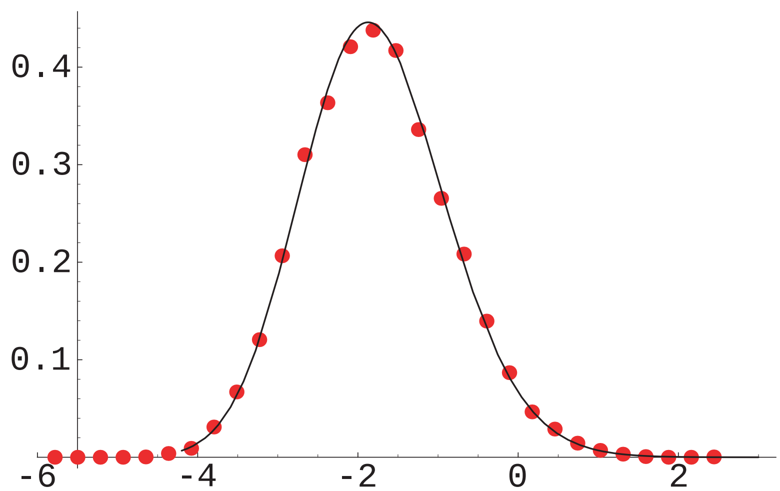

Figure 8: The Tracy-Widom distribution F together with Histogram of 10^4 simulated values of $n^{1/3}(L_n/\sqrt{n}-2)$ with $n = 3 \times 10^7$.

7 Part 5: Combining the Monte-Carlo results of Section 3 with Odlyzko and Rains

Given the Monte-Carlo results of Odlyzko and Rains (2000) posted at

`http://www.dtc.umn.edu/~odlyzko/tables/index.html`

it seems reasonable to combine with the results presented here in Section 3. Plots 10 - 13 below parallel the plots 3-6 in Section 3, but with the mean and variance summaries from the experiments of Odlyzko and Rains (2000) added. Note that all the black dots (Wellner) are based on 10^4 Monte-Carlo replications; the red dots (Odlyzko-Rains) are based on 10^7 replications for $n = 10^4$, 6×10^5 replications for $n = 10^5$, 10^5 replications for $n = 10^6$ and $n = 10^7$, 10^4 replications for $n = 10^8$, 2×10^3 replications for $n = 10^9$, and 4×10^3 replications for $n = 10^{10}$.

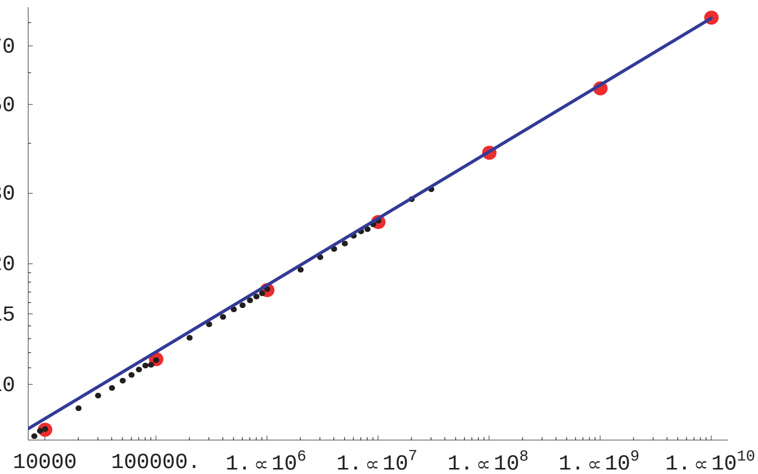

Figure 9: Empirical means of $(2\sqrt{n} - L_n)$ and line with slope 1/6.

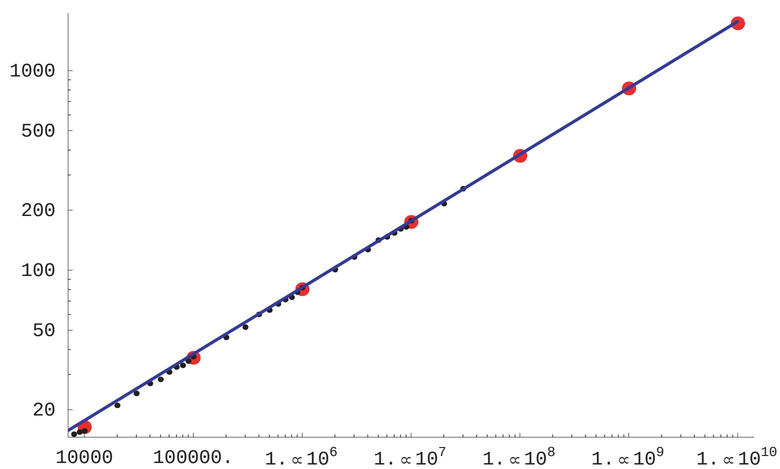

Figure 10: Empirical variances of $(2\sqrt{n} - L_n)$ and line with slope 1/3.

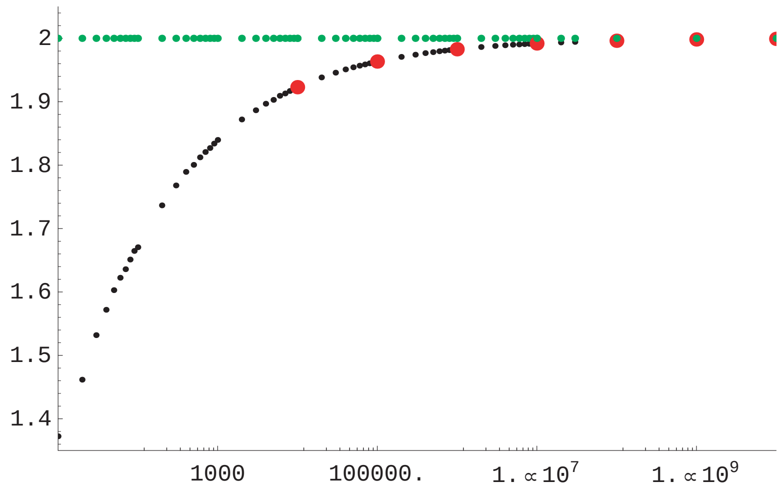

Figure 11: Empirical means of $L_n/\sqrt{n}$.

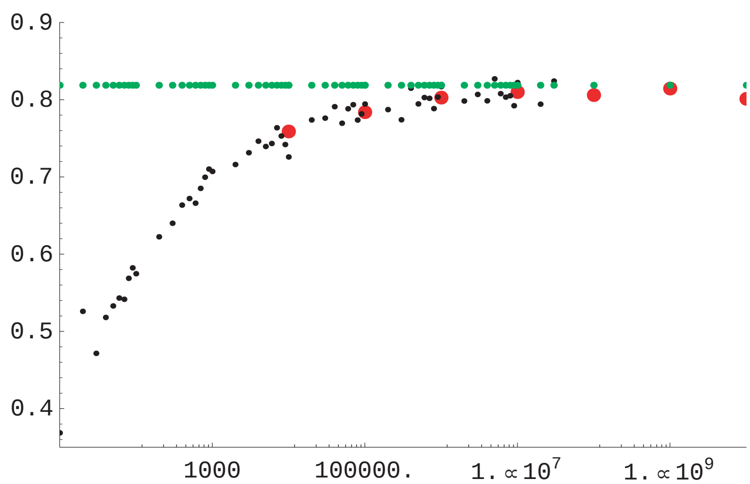

Figure 12: Empirical variances of $n^{1/3}(L_n/\sqrt{n} - 2)$.

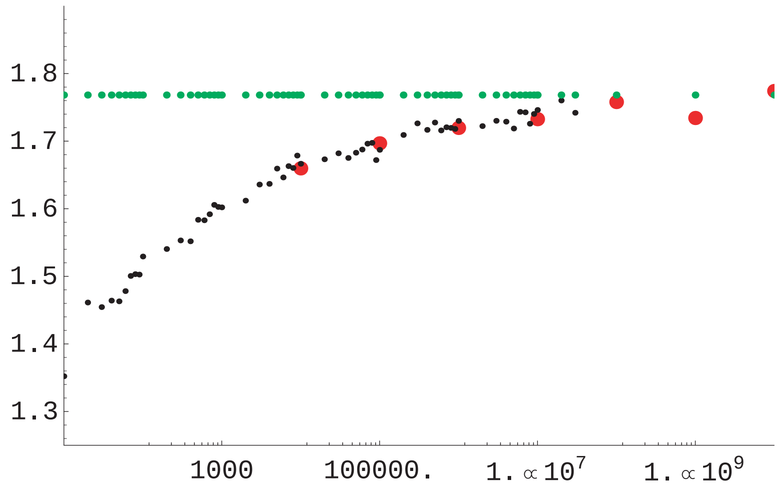

Figure 13: Empirical means of $n^{1/3}(2 - L_n/\sqrt{n})$.

Acknowledgements: Thanks are due to Persi Diaconis for a long conversation in August 1997 about this problem. I owe thanks to Greg Warnes for obtaining the C program from David Eppstein, for getting it working, and for time on the machines at the Fred Hutchinson Cancer Research Center. I also owe thanks to Craig Tracy for providing the Mathematica program used to calculate the Tracy-Widom distribution shown in Figure 8.

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University of Washington
Statistics
Box 354322
Seattle, Washington 98195-4322
U.S.A.
e-mail: jaw@stat.washington.edu