Nonparametric estimation under Shape Restrictions

Jon A. Wellner

University of Washington, Seattle

Statistical Seminar, Frejus, France

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Outline: Five Lectures on Shape Restrictions

- L1: Monotone functions: maximum likelihood and least squares
- L2: Optimality of the MLE of a monotone density (and comparisons?)
- **L3: Estimation of convex and $k-$monotone density functions**
- L4: Estimation of log-concave densities: $d = 1$ and beyond
- L5: More on higher dimensions and some open problems

Outline: Lecture 3

- A: Convex decreasing and $k-$monotone densities as mixtures
- B: Existence and uniqueness of MLE, $k-$monotone, $k \geq 2$
- C: Consistency, $k-$monotone, $2 \leq k \leq \infty$
- D: Global rates of convergence: $2 \leq k < \infty$
- E: Local rate of convergence: $k = 2$
- F: Limiting distributions at a fixed point: $k = 2$

A. Convex decreasing and $k-$monotone densities as mixtures

Definition 1. Let k be an integer, $k \geq 2$. A density f on $\mathbb{R}^+$ is said to be $k-$*monotone* if $f^{(j)}$ exists for $j = 1, \ldots, k-2$ with $(-1)^j f^{(j)}(x) \geq 0$ and $f^{(k-2)}$ is convex. Let $\mathcal{D}_k$ denote the class of all $k-$monotone densities

Definition 2. A density f on $\mathbb{R}^+$ is said to be *completely monotone* if $f^{(j)}$ exists for $j = 1, \ldots$ with $(-1)^j f^{(j)}(x) \geq 0$ for all j. Let $\mathcal{D}_\infty$ denote the class of all completely monotone densities.

In part D of Lecture 2 it was noted that every monotone decreasing density f on R^+ is a scale mixture of uniform densities, and every decreasing convex density f is a scale mixture of triangular (or Beta$(1,2)$) densities. In fact this extends to the class of $k-$monotone densities.

A. Convex decreasing and $k-$monotone densities as mixtures

$f \in \mathcal{D}_k$ if and only if

$$f(x) = \int_0^\infty k y^{-1} (1 - x/y)_+^{k-1} dG(y) \quad \text{for some distribution } G.$$

while $f \in \mathcal{D}_\infty$ if and only if

$$f(x) = \int_0^\infty y^{-1} \text{exp}(-x/y) dG(y) \quad \text{for some distribution } G.$$

It is convenient to recast this as follows:

Proposition 1. (Williamson, 1956; Lévy, 1962; Bernstein) A density f is a $k-$monotone (completely monotone) density if and only if it can be represented as a scale mixture of Beta$(1, k)$ (exponential) densities; i.e. with $x_+ \equiv x 1\{x \geq 0\}$,

$$f(x) = \begin{cases} \int_0^\infty y^{-1} \left(1 - \frac{x}{ky}\right)_+^{k-1} dG(y), & k \in \{1, 2, \ldots\}, \\ \int_0^\infty y^{-1} \text{exp}(-x/y) dG(y), & k = \infty \,, \end{cases} \tag{1}$$

for some distribution function G on $(0, \infty)$.

A. Convex decreasing and $k-$monotone densities as mixtures

The inversion formulas corresponding to these mixture representations are given in the following proposition.

Proposition 2. Suppose that f is a $k-$monotone density with distribution function F (so $F(x) = \int_0^x f(t)dt$). Then the distribution function $G = G_k$ of (**??**) is given at continuity points of G_k by

$$G_k(t) = \sum_{j=0}^{k} \frac{(-1)^j}{j!} (kt)^j F^{(j)}(kt)\,, \qquad (2)$$

and the distribution function $G = G_\infty$ of the $k = \infty$ part of (**??**) is given at continuity points of G_∞ by

$$G_\infty(t) = \lim_{k\to\infty} G_k(t)\,. \qquad (3)$$

A. Convex decreasing and $k-$monotone densities as mixtures

It will be convenient to have notation for the classes of functions given by the mixing representations in (**??**) when the mixing measure G is not require to have mass 1, and hence the resulting functions f are not necessarily densities. We denote these classes by $\mathcal{M}_k$ for $1 \leq k \leq \infty$.

B. Existence and uniqueness of MLE, $k-$monotone, $k \geq 2$

Now suppose that $X_1, \ldots, X_n$ are i.i.d. $f_0 \in \mathcal{D}_k$ for some $k \in \{2, \ldots, \infty\}$. The MLE $\widehat{f}_n \equiv \widehat{f}_{n,k}$ is defined by

$$\widehat{f}_n = \text{argmax}\{\mathbb{P}_n \log f : f \in \mathcal{D}_k\}.$$

The LSE $\widetilde{f}_n \equiv \widetilde{f}_{n,k}$ of f_0 is defined by

$$\widetilde{f}_n \equiv \text{argmin}\{\psi_n(f) : \ f \in \mathcal{M}_k \cap L_2(\lambda)\}$$

where

$$\psi_n(f) \equiv \frac{1}{2}\int_0^\infty f^2(x)dx - \int_0^\infty f(x)d\mathbb{F}_n(x).$$

B. Existence and uniqueness of MLE, $k-$monotone, $k \geq 2$

Theorem. ($k = 2$: Groeneboom, Jongbloed, W (2001);
$2 < k < \infty$: Balabdaoui (2004);
$2 \leq k < \infty$: Seregin (2010);
$k = \infty$: Jewell (1982))

(a) For $2 \leq k \leq \infty$ the MLE $\widehat{f}_n$ exists and is unique.

(b) For $2 \leq k < \infty$ the LSE $\widetilde{f}_n$ exists and is unique.

(c) $\widetilde{f}_{n,k} \neq \widehat{f}_{n,k}$ for all $k \geq 2$.

Proof. Methods:

- Nonparametric estimation in mixtures: Lindsay (1983a,b); Lindsay (1995); Lindsay and Roeder (1993).
- Positivity / total positivity: Schoenberg and Whitney (1953); Polya and Szëgo (1925); Karlin (1968).

B. Existence and uniqueness of MLE, $k-$monotone, $k \geq 2$

Theorem. $\widehat{f}_{n,k}$ is characterized by:
(a) $2 \leq k < \infty$: The "Fenchel" conditions hold:

$$\int_0^\infty \frac{k(y-x)_+^{k-1}}{y^k \widehat{f}_{n,k}(x)} d\mathbb{F}_n(x) \leq 1 \quad \text{for all } y > 0$$

with equality if and only if $y \in \text{supp}(\widehat{G}_{n,k})$.
(b) $k = \infty$: The "Fenchel" conditions hold:

$$\int_0^\infty \frac{\exp(-x/y)}{y \widehat{f}_{n,\infty}(x)} d\mathbb{F}_n(x) \leq 1 \quad \text{for all } y > 0$$

with equality if and only if $y \in \text{supp}(\widehat{G}_{n,\infty})$.

B. Existence and uniqueness of MLE, $k-$monotone, $k \geq 2$

To state the characterization of the LSE $\widetilde{f}_n$ we define $\mathbb{Y}_{n,k}$ and $\widetilde{\mathbb{H}}_{n,k}$ by:

$$\mathbb{Y}_{n,k}(x) \equiv \int_0^x \int_0^{x_{k-1}} \cdots \int_0^{x_2} \mathbb{F}_n(x_1) dx_1 dx_2 \cdots dx_{k-1},$$
$$\widetilde{\mathbb{H}}_{n,k}(x) \equiv \int_0^x \int_0^{x_{k-1}} \cdots \int_0^{x_2} \int_0^{x_1} \widetilde{f}_{n,k}(x_0) dx_0 dx_2 \cdots dx_{k-1}$$

for $x \geq 0$.

Theorem. (a) $\widetilde{f}_{n,k}$ is characterized by:

$$\widetilde{\mathbb{H}}_{n,k}(x) \geq \mathbb{Y}_{n,k}(x) \quad \text{for all} \quad x \geq 0. \tag{4}$$

with equality holding if and only if $x \in \text{supp}(\widetilde{G}_{n,k})$.
(b) The equality conditions can be expressed as

$$\int_0^\infty (\widetilde{\mathbb{H}}_{n,k}(y) - \mathbb{Y}_{n,k}(y)) d\widetilde{\mathbb{H}}_{n,k}^{(2k-1)}(y) = 0.$$

B. Existence and uniqueness of MLE, $k-$monotone, $k \geq 2$

It is not hard to see that

$$\mathbb{Y}_{n,k}(y) = \int_0^y \frac{(y-x)^{k-1}}{(k-1)!} d\mathbb{F}_n(x) = \int_0^\infty \frac{(y-x)_+^{k-1}}{(k-1)!} d\mathbb{F}_n(x),$$

$$\widetilde{\mathbb{H}}_{n,k}(y) = \int_0^y \frac{(y-x)^{k-1}}{(k-1)!} d\widetilde{F}_{n,k}(x) = \int_0^\infty \frac{(y-x)_+^{k-1}}{(k-1)!} d\widetilde{F}_{n,k}(x)$$

Thus the inequality part of the second theorem can be rewritten as

$$\int_0^\infty \frac{(y-x)_+^{k-1}}{(k-1)!} d\left(\widetilde{F}_{n,k}(x) - \mathbb{F}_n(x)\right) \geq 0 \quad \text{for all } y > 0.$$

Similarly, the equality part of the first theorem can be rewritten as

$$\int_0^\infty \frac{k(y-x)_+^{k-1}}{y^k \widehat{f}_{n,k}(x)} d\left(\widehat{F}_{n,k}(x) - \mathbb{F}_n(x)\right) \geq 0 \quad \text{for all } y > 0.$$

C: Consistency, $k-$monotone, $2 \leq k \leq \infty$

Suppose that $X_1, \ldots, X_n$ are i.i.d. P_0 with density $p_0 \in \mathcal{P}$, a convex class of densities with respect to a $\sigma-$finite measure μ on a measurable space $(\mathcal{X}, \mathcal{A})$. Let

$$\widehat{p}_n \equiv \text{argmax}_{p \in \mathcal{P}} \mathbb{P}_n \text{log}(p) \, .$$

For $0 < \alpha \leq 1$, let $\varphi_\alpha(t) = (t^\alpha - 1)/(t^\alpha + 1)$ for $t \geq 0$, $\varphi_\alpha(t) = -1$ for $t < 0$. Then φ_α is bounded and continuous for each $\alpha \in (0, 1]$. For $0 < \beta < 1$ define

$$h_\beta^2(p, q) \equiv 1 - \int p^\beta q^{1-\beta} d\mu \, .$$

Note that $h_{1/2}(p, q) \equiv H(p, q)$ is the Hellinger distance between p and q, and by Hölder's inequality, $h_\beta(p, q) \geq 0$ with equality if and only if $p = q$ a.e. μ.

C: Consistency, $k-$monotone, $2 \le k \le \infty$

Proposition: (Pfanzagl; van de Geer) Suppose that $\mathcal{P}$ is convex. Then

$$h^2_{1-\alpha/2}(\widehat{p}_n, p_0) \le (\mathbb{P}_n - P_0)\left(\varphi_\alpha\left(\frac{\widehat{p}_n}{p_0}\right)\right) .$$

In particular, when $\alpha = 1$ we have, with $\varphi \equiv \varphi_1$,

$$H^2(\widehat{p}_n, p_0) \le (\mathbb{P}_n - P_0)\left(\varphi\left(\frac{\widehat{p}_n}{p_0}\right)\right) = (\mathbb{P}_n - P_0)\left(\frac{2\widehat{p}_n}{\widehat{p}_n + p_0}\right) .$$

Corollary: (Pfanzagl (1988); van de Geer, (1993, 1996)) Suppose that $\{\varphi_\alpha(p/p_0) : \ p \in \mathcal{P}\}$ is a P_0 Glivenko-Cantelli class. Then for each $0 < \alpha \le 1$, $h_{1-\alpha/2}(\widehat{p}_n, p_0) \to_{a.s.} 0$.

C: Consistency, $k-$monotone, $2 \leq k \leq \infty$

Proof. Since $\mathcal{P}$ is convex and $\widehat{p}_n$ maximizes $\mathbb{P}_n \log p$ over $\mathcal{P}$, it follows that

$$\mathbb{P}_n \log \frac{\widehat{p}_n}{(1-t)\widehat{p}_n + t p_1} \geq 0$$

for all $0 \leq t \leq 1$ and every $p_1 \in \mathcal{P}$; this holds in particular for $p_1 = p_0$. Note that equality holds if $t = 0$. Differentiation of the left side with respect to t at $t = 0$ yields

$$\mathbb{P}_n \frac{p_1}{\widehat{p}_n} \leq 1 \qquad \text{for every } \; p_1 \in \mathcal{P} \, .$$

If $L : (0, \infty) \mapsto R$ is increasing and $t \mapsto L(1/t)$ is convex, then Jensen's inequality yields

$$\mathbb{P}_n L\left(\frac{\widehat{p}_n}{p_1}\right) \geq L\left(\frac{1}{\mathbb{P}_n(p_1/\widehat{p}_n)}\right) \geq L(1) = \mathbb{P}_n L\left(\frac{p_1}{p_1}\right) \, .$$

Choosing $L = \varphi_\alpha$ and $p_1 = p_0$ in this last inequality and noting that $L(1) = 0$, it follows that

$$0 \leq \mathbb{P}_n \varphi_\alpha(\widehat{p}_n/p_0) = (\mathbb{P}_n - P_0)\varphi_\alpha(\widehat{p}_n/p_0) + P_0 \varphi_\alpha(\widehat{p}_n/p_0) \, ; \qquad (5)$$

C: Consistency, $k-$monotone, $2 \leq k \leq \infty$

see van der Vaart and Wellner (1996) page 330, and Pfanzagl (1988), pages 141 - 143. Now we show that

$$P_0\varphi_\alpha(p/p_0) = \int \frac{p^\alpha - p_0^\alpha}{p^\alpha + p_0^\alpha} dP_0 \leq -\left(1 - \int p_0^\beta p^{1-\beta} d\mu\right) \qquad (6)$$

for $\beta = 1 - \alpha/2$. Note that this holds if and only if

$$-1 + 2\int \frac{p^\alpha}{p_0^\alpha + p^\alpha} p_0 d\mu \leq -1 + \int p_0^\beta p^{1-\beta} d\mu \,,$$

or

$$\int p_0^\beta p^{1-\beta} d\mu \geq 2\int \frac{p^\alpha}{p_0^\alpha + p^\alpha} p_0 d\mu \,.$$

But his holds if

$$p_0^\beta p^{1-\beta} \geq 2\frac{p^\alpha p_0}{p_0^\alpha + p^\alpha} \,.$$

C: Consistency, $k-$monotone, $2 \leq k \leq \infty$

With $\beta = 1 - \alpha/2$, this becomes

$$\frac{1}{2}(p_0^{\alpha} + p^{\alpha}) \geq {p_0^{\alpha/2}} p^{\alpha/2} = \sqrt{p_0^{\alpha} p^{\alpha}}\,,$$

and this holds by the arithmetic mean - geometric mean inequality. Thus (**??**) holds. Combining (**??**) with (**??**) yields the claim of the proposition. The corollary follows by noting that $\varphi(t) = (t-1)/(t+1) = 2t/(t+1) - 1$.

C: Consistency, $k-$monotone, $2 \le k \le \infty$

To apply this to the MLEs $\widehat{f}_{n,k} \in \mathcal{D}_k$, we take $\mathcal{P} = \mathcal{D}_k$, which is convex in view of the mixture representation.

We first show that the map $G \mapsto f_G(x)$ is continuous with respect to the topology of vague convergence for distributions G. This follows easily since for each fixed $x > 0$ the kernels

$$y \mapsto k y^{-1}(1 - x/y)_+^{k-1} \equiv m_k(x, y)$$

for this mixing family are bounded, continuous, and satisfy $m_k(x, y) \to 0$ as $y \to 0$ or ∞ for every $x > 0$. Since vague convergence of distribution functions implies that integrals of bounded continuous functions vanishing at infinity converge, it follows that $G \mapsto f_G(x)$ is continuous with respect to the vague topology for every $x > 0$. This implies, that the family

$$\mathcal{F}_k = \left\{ \frac{f_G}{f_G + f_0} : \ G \ \text{ a d.f. on } \mathbb{R}^+ \right\}$$

C: Consistency, $k-$monotone, $2 \le k \le \infty$

is pointwise, for a.e. x, continuous in G wrt the vague topology. Since the family of sub-distribution functions G on $\mathbb{R}$ is compact for the vague topology (Bauer (1972), p. 241), and the family of functions $\mathcal{F}_k$ is uniformly bounded by 1, we conclude from the argument of Wald (1949) that

$$N_{[\,]}(\epsilon, \mathcal{F}_k, L_1(P_0)) < \infty \quad \text{for every } \epsilon > 0.$$

Thus $\mathcal{F}_k$ is $P0$- Glivenko-Cantelli and we conclude that $\widehat{f}_{n,k} = f_{\widehat{G}_n}$ satifies

$$H(\widehat{f}_{n,k}, f_0) \to_{a.s.} 0\,.$$

The same argument works for $k = \infty$ and yields a different proof of a result of Jewell (1982).

C: Consistency, $k-$monotone, $2 \leq k \leq \infty$

Based on the bound

$$f(x) \leq \frac{1}{x}\left(1 - \frac{1}{k}\right)^{k-1} \quad \text{for all } x > 0, \quad f \in \mathcal{D}_k$$

and subsequence arguments, it follows that for each $c > 0$

$$\sup_{x \geq c} |\widehat{f}_{n,k}(x) - f_{0,k}(x)| \to 0 \quad \text{as} \quad n \to \infty,$$

$$\sup_{x \geq c} |\widehat{f}^{(j)}_{n,k}(x) - f^{(j)}_{0,k}(x)| \to 0 \quad \text{as} \quad n \to \infty, 1 \leq j \leq k-1, \quad \text{and}$$

$$\widehat{f}^{(k-1)}_n(x) \to_{a.s.} f^{(k-1)}_{0,k}(x) \quad \text{if the derivative} \quad f^{(k-1)}_{0,k}(x) \text{ exists.}$$

What about rates of convergence?

D: Global Rates, $2 \leq k < \infty$

Based on:

- Empirical process fluctuation bound:
  Birgé & Massart; van der Vaart & W
- Rate of convergence result:
  Birgé & Massart; van der Vaart & W (1996)
- Entropy bound for bounded sub-classes of $\mathcal{D}_k$:
  Gao & W (2009)
- If $f_0(0) < \infty$, then $\widehat{f}_{n,k}(0) = O_p(1)$. Gao & W (2009)

D: Global Rates, $2 \le k < \infty$

Empirical process result:

Suppose that $\mathcal{P}$ is a collection of densities, $\mathcal{P}_0 \subset \mathcal{P}$

Theorem. (Thm 3.2.5, vdV & W, simplified) Suppose that $X_1, \ldots, X_n$ are i.i.d. P_0 with density $p_0 \in \mathcal{P}_0$. Let H be the Hellinger distance between densities, and let m_p be defined, for $p \in \mathcal{P}$, by

$$m_p(x) = \log\left((p(x) + p_0(x))/(2p_0(x))\right) .$$

Then $\mathbb{M}(p) - \mathbb{M}(p_0) \equiv P_0(m_p - m_{p_0}) \lesssim -H^2(p, p_0)$. Furthermore, with $\mathcal{M}_\delta = \{m_p - m_{p_0} : \; H(p, p_0) \le \delta, \; p \in \mathcal{P}_0\}$, we also have

$$E^*_{P_0} \|\mathbb{G}_n\|_{\mathcal{M}_\delta} \lesssim \tilde{J}_{[\,]}(\delta, \mathcal{P}_0, H) \left(1 + \frac{\tilde{J}_{[\,]}(\delta, \mathcal{P}_0, H)}{\delta^2 \sqrt{n}}\right) \equiv \phi_n(\delta, \mathcal{P}_0), (7)$$

where

$$\tilde{J}_{[\,]}(\delta, \mathcal{P}_0, H) = \int_0^\delta \sqrt{1 + \log N_{[\,]}(\epsilon, \mathcal{P}_0, H)}\, d\epsilon.$$

D: Global Rates, $2 \le k < \infty$

Entropy bound for bounded sub-classes of $\mathcal{D}_k$: Let

$$\mathcal{P}_0 \equiv \mathcal{D}_k^B([0,A]) \equiv \{f \in \mathcal{D}_k : \ f(0) \le B, \ f(x) = 0 \ \text{ if } x > A\}.$$

Gao & W (2009) show that for $\epsilon > 0$

$$\log N_{[\cdot]}(\epsilon, \mathcal{D}_k^B([0,A]), H) \le C\epsilon^{-1/k}$$

where $C = C_k(A,B)$.

If $f_0(0) < \infty$, then $\widehat{f}_{n,k}(0) = O_p(1)$. By the characterization of $\widehat{f}_{n,k}$,

$$1 \ \ge \ \int_0^y \frac{k}{y^k} \frac{(y-x)^{k-1}}{\widehat{f}_{n,k}(x)} d\mathbb{F}_n(x) \ \text{ for all } y > 0$$

with equality if $y \in \{\tau_1, \ldots, \tau_m\} \equiv \text{supp}(\widehat{G}_{n,k})$ where $0 < \tau_1 < \cdots < \tau_m < \infty$. Thus for $y = \tau_1$ and $0 \le x \le \tau_1$,

$$1 = \frac{k}{\tau_1} \int_0^{\tau_1} \frac{(1 - x/\tau_1)^{k-1}}{\widehat{f}_{n,k}(x)} d\mathbb{F}_n(x)$$

where

$$\begin{aligned}\widehat{f}_{n,k}(x) &= \int_0^\infty \frac{k}{y}\left(1-\frac{x}{y}\right)_+^{k-1} d\widehat{G}_{n,k}(y) \\ &\ge \left(1-\frac{x}{\tau_1}\right)_+^{k-1}\int_0^\infty \frac{k}{y} d\widehat{G}_{n,k}(y) = (1-x/\tau_1)_+^{k-1}\widehat{f}_{n,k}(0).\end{aligned}$$

Hence

$$1 \le \frac{k}{\tau_1}\int_0^{\tau_1}\frac{(1-x/\tau_1)^{k-1}}{\widehat{f}_{n,k}(0)(1-x/\tau_1)^{k-1}} d\mathbb{F}_n(x) = \frac{k}{\tau_1\widehat{f}_{n,k}(0)}\mathbb{F}_n(\tau_1),$$

which yields

$$\begin{aligned}\widehat{f}_{n,k}(0) &\le k\frac{\mathbb{F}_n(\tau_1)}{\tau_1} \le k\sup_{t>0}\frac{\mathbb{F}_n(t)}{t} \\ &\le k\sup_{t>0}\frac{\mathbb{F}_n(t)}{F_0(t)} f_0(0) = O_p(1).\end{aligned}$$

D: Global Rates, $2 \leq k < \infty$

Combining these facts proves:

Theorem. (Gao & W, 2009) Suppose that $f_0 \in \mathcal{D}_k^B([0, A])$ for some $0 < A, B < \infty$. Then $\{\widehat{f}_{n,k}\}$ satisfies

$$H(\widehat{f}_{n,k}, f_0) = O_p(n^{-\frac{k}{2k+1}}).$$

Questions:

- What is the rate for $\widehat{f}_{n,\infty} \in \mathcal{D}_\infty$?
- Can we go beyond $\mathcal{D}_k^B([0, A])$?
- Is $n^{-1/(2k+1)}$ the rate of convergence of $d_{BL}(\widehat{G}_{n,k}, G_0)$?

E: Rates of convergence: local results, $k = 2$

- Difficulty: no switching relation! Study LSE as first step.
- Proceed by localizing the Fenchel conditions
  - ▷ Step 1: localization rate or tightness result
    Empirical process theory: Kim-Pollard type lemmas
  - ▷ Step 2: Weak convergence of the localized *driving process* to a limit Gaussian *driving process*
    Empirical process theory: bracketing CLT with functions dependent on n.
  - ▷ Step 3: Preservation of (localized) characterizing relations in the limit.
  - ▷ Step 4: Establishing uniqueness of the limiting (Gaussian world) estimator resulting from the Fenchel relations.
- Cross check limit distributions with lower bound theory.

E: Rates of convergence: local results, $k = 2$

Step 1: Localization:

- Fenchel characterization implies midpoint properties.
- Midpoint properties + Kim-Pollard type lemma implies gap rate.
- Gap rate $\tau_n^+ - \tau_n^- = O_p(n^{-1/5})$ yields tightness.

E: Rates of convergence: local results, $k = 2$

Mid-point properties: Recall the Fenchel characterization of the LSE, $k = 2$:

$$\widetilde{\mathbb{H}}_n(x) \geq \mathbb{Y}_n(x) \quad \text{for all} \quad x \geq 0. \tag{8}$$

with equality holding if and only if $x \in \text{supp}(\widetilde{G}_n)$.

(b) The equality conditions can be expressed as

$$\int_0^\infty \left(\widetilde{\mathbb{H}}_n(y) - \mathbb{Y}_n(y) \right) d\widetilde{\mathbb{H}}_n^{(3)}(y) = 0.$$

It follows that $\widetilde{\mathbb{H}}_n$ is piecewise cubic: for $\tau_1 < \tau_2$, with $\tau_1, \tau_2 \in \text{supp}(\widetilde{G}_{n,2})$ two successive touch points,

$$\widetilde{\mathbb{H}}_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad \text{on} \quad [\tau_1, \tau_2]$$

where a_0, a_1, a_2, a_3 are determined by

$$\widetilde{\mathbb{H}}_n(\tau_j) = \mathbb{Y}_n(\tau_j), \quad j = 1, 2, \quad \text{and}$$
$$\widetilde{F}_n(\tau_j) = \mathbb{F}_n(\tau_j), \quad j = 1, 2.$$

E: Rates of convergence: local results, $k = 2$

Upshot: for $x \in [\tau_1, \tau_2]$

$$\begin{aligned}\widetilde{\mathbb{H}}_n(x) &= \{\mathbb{Y}_n(\tau_2)(x-\tau_1) + \mathbb{Y}_n(\tau_1)(\tau_2 - x)\}/\Delta\tau \\ &\quad -\frac{1}{2}\left\{\frac{\Delta\mathbb{F}_n}{\Delta\tau} + \frac{4(\overline{\mathbb{F}}_n\Delta\tau - \Delta\mathbb{Y}_n)(x-\overline{\tau})}{(\Delta\tau)^3}\right\}(x-\tau_1)(x-\tau_2),\end{aligned}$$

so, with $\overline{\tau} \equiv (\tau_2 + \tau_1)/2$ and $\Delta\tau \equiv \tau_2 - \tau_1$,

$$\widetilde{\mathbb{H}}_n(\overline{\tau}) = \overline{\mathbb{Y}}_n - \frac{1}{8}\Delta\mathbb{F}_n\Delta\tau$$

where

$$\begin{aligned}\Delta\mathbb{Y}_n &\equiv \mathbb{Y}_n(\tau_2) - \mathbb{Y}_n(\tau_1), \qquad \Delta\mathbb{F}_n \equiv \mathbb{F}_n(\tau_2) - \mathbb{F}_n(\tau_1), \\ \overline{\mathbb{Y}}_n &\equiv (\mathbb{Y}_n(\tau_2) + \mathbb{Y}_n(\tau_1))/2, \quad \overline{\mathbb{F}}_n \equiv (\mathbb{F}_n(\tau_2) + \mathbb{F}_n(\tau_1))/2.\end{aligned}$$

Now we can rewrite $\widetilde{\mathbb{H}}_n(\overline{\tau}) \geq \mathbb{Y}_n(\overline{\tau})$ as

$$\overline{\mathbb{Y}}_n - \frac{1}{8}\Delta\mathbb{F}_n\Delta\tau \geq \mathbb{Y}_n(\overline{\tau}),$$

E: Rates of convergence: local results, $k = 2$

Now let x_0 with $f_0^{(2)}(x_0) > 0$ be fixed, let $\xi_n \to x_0$, and take

$$\tau_1 \equiv \tau_n^- \equiv \max\{t \in \text{supp}(\tilde{G}_n) : \ t \le \xi_n\},$$
$$\tau_2 \equiv \tau_n^+ \equiv \min\{t \in \text{supp}(\tilde{G}_n) : \ t > \xi_n\}.$$

Then $\tilde{\mathbb{H}}_n(\bar{\tau}_n) \ge \mathbb{Y}_n(\bar{\tau}_n)$ can be rewritten as

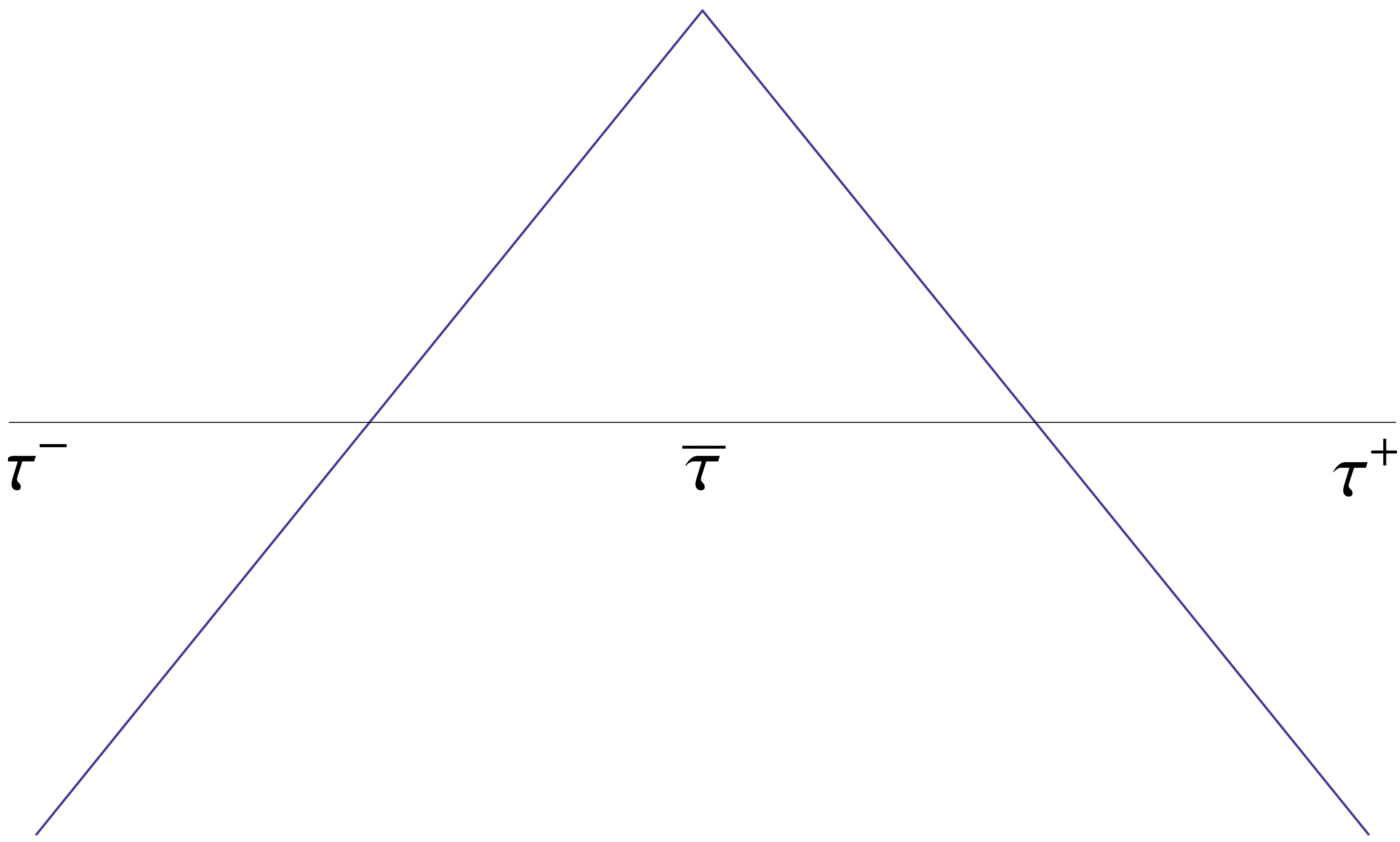
$$\frac{1}{2}\left(\mathbb{Y}_n(\tau_n^+) + \mathbb{Y}_n(\tau_n^-)\right) - \frac{1}{8}\left\{\mathbb{F}_n(\tau_n^+) - \mathbb{F}_n(\tau_n^-)\right\}(\tau_n^+ - \tau_n^-) \ge \mathbb{Y}_n(\bar{\tau}_n). \quad (9)$$

Replacing $\mathbb{Y}_n$ and $\mathbb{F}_n$ by their deterministic counterparts and then expanding the integrands at $\bar{\tau}_n$ yields

$$\begin{aligned}
&\int_{\bar{\tau}_n}^{\tau_n^+} (\tau_n^+ - x) f_0(x)dx + \int_{\tau_n^-}^{\bar{\tau}_n} (x - \tau_n^-) f_0(x)dx - \frac{1}{4}(\tau_n^+ - \tau_n^-) \int_{\tau_n^-}^{\tau_n^+} f_0(x)dx \\
&= \int_{[\bar{\tau}_n, \tau_n^+]} \{\frac{1}{2}(\bar{\tau}_n + \tau_n^+) - x\} f_0(x)dx + \int_{[\tau_n^-, \bar{\tau}_n]} \{x - \frac{1}{2}(\tau_n^- + \bar{\tau}_n)\} f_0(x)dx \\
&= -\frac{1}{192} f_0(\bar{\tau}_n)(\tau_n^+ - \tau_n^-)^4 + o_p(\tau_n^+ - \tau_n^-)^4,
\end{aligned}$$

by using consistency of $\tilde{f}_n$ to ensure that $\bar{\tau}_n$ belongs to a sufficiently small neighborhood of x_0.

E: Rates of convergence: local results, $k = 2$



E: Rates of convergence: local results, $k = 2$

The difference between (**??**) and the deterministic version is

$$\begin{aligned}
&\int_{[\tau_n^-,\overline{\tau}_n]} (z - (\tau_n^- + \overline{\tau}_n)/2) d\left(\mathbb{F}_n(z) - F_0(z)\right) \\
&\quad + \int_{[\overline{\tau}_n,\tau_n^+]} ((\tau_n^+ + \overline{\tau}_n)/2 - z) d\left(\mathbb{F}_n(z) - F_0(z)\right) \\
&\equiv U_n(\tau_n^-, \overline{\tau}_n) - U_n(\overline{\tau}_n, \tau_n^+) \quad \text{where}
\end{aligned}$$

$$U_n(x,y) \equiv \int_{[x,y]} (z - (x+y)/2) d(\mathbb{F}_n(z) - F_0(z)).$$

By an empirical process argument – as in Kim and Pollard (1991), there exist constants $\delta > 0$ and $c_0 > 0$ such that, for each $\epsilon > 0$ and each x satisfying $|x - x_0| < \delta$,

$$|U_n(x,y)| \le \epsilon |y - x|^4 + O_p(n^{-4/5}), \quad \text{for all} \quad 0 \le y - x \le c_0.$$

This implies that

$$|U_n(\tau_n^-, \overline{\tau}_n) - U_n(\overline{\tau}_n, \tau_n^+)| \le \epsilon(\tau_n^+ - \tau_n^-)^4 + O_p(n^{-4/5}).$$

E: Rates of convergence: local results, $k = 2$

Putting the pieces together by choosing $\epsilon = f_0^{(2)}(x_0)/384$ it follows that

$$
\begin{aligned}
&-\frac{1}{192} f_0^{(2)}(x_0)(\tau_n^+ - \tau_n^-)^4 + o_p(\tau_n^+ - \tau_n^-)^4 \\
&\quad + \frac{1}{384} f_0^{(2)}(x_0)(\tau_n^+ - \tau_n^-)^4 + O_p(n^{-4/5}) \geq 0,
\end{aligned}
$$

and hence

$$
\tau_n^+ - \tau_n^- = O_p(n^{-1/5}).
$$

This leads to:

Proposition: Suppose that $f_0'(x_0) < 0$, $f_0^{(2)}(x_0) > 0$ and $f_0^{(2)}$ continuous in a neighborhood of x_0. Then

$$
\sup_{|t| \leq M} |\widetilde{f}_n(x_0 + n^{-1/5}t) - f_0(x_0) - n^{-1/5} t f_0'(x_0)| = O_p(n^{-2/5}),
$$

$$
\sup_{|t| \leq M} |\widetilde{f}_n'(x_0 + n^{-1/5}t) - f_0'(x_0)| = O_p(n^{-2/5}).
$$

... and a corresponding result for the MLE $\widehat{f}_n$.

F: Limiting distributions at a fixed point: $k = 2$

Step 2: Localize the Fenchel conditions

Define

$$\begin{aligned}
\mathbb{Y}_n^{loc}(t) &\equiv n^{4/5} \int_{x_0}^{x_0+n^{-1/5}t} \Big\{ \mathbb{F}_n(v) - \mathbb{F}_n(x_0) \\
&\qquad - \int_{x_0}^{v} (f_0(x_0) + (u - x_0) f_0'(x_0)) du \Big\} dv \\
&\stackrel{d}{=} n^{3/10} \int_{x_0}^{x_0+n^{-1/5}} \{ \mathbb{U}_n(F_0(v)) - \mathbb{U}_n(F(x_0)) \} dv \\
&\qquad + \frac{f_0^{(2)}(x_0)}{4!} t^4 + o(1) \\
&\rightsquigarrow \sqrt{f_0(x_0)} \int_0^t W(s) ds + \frac{f_0^{(2)}(x_0)}{4!} t^4 \quad \text{by KMT} \\
&\qquad \text{or by theorem 2.11.22 or 2.11.23, vdV \& W (1996)} \\
&\equiv a \int_0^t W(s) ds + bt^4 \equiv \mathbb{Y}_{a,b}(t).
\end{aligned}$$

F: Limiting distributions at a fixed point: $k = 2$

Similarly, define

$$\widetilde{\mathbb{H}}_n^{loc}(t) \equiv n^{4/5} \int_{x_0}^{x_0+n^{-1/5}t} \int_{x_0}^{v} \{\widetilde{f}_n(u) - f_0(x_0) - (u - x_0) f_0'(x_0)\} du dv + \widetilde{B}_n t + \widetilde{A}_n$$

where

$$\widetilde{A}_n \equiv n^{4/5}(\widetilde{\mathbb{H}}_n(x_0) - \mathbb{Y}_n(x_0)) = O_p(1)$$
$$\widetilde{B}_n \equiv n^{3/5}(\widetilde{F}_n(x_0) - \mathbb{F}_n(x_0)) = O_p(1).$$

Furthermore

$$\widetilde{\mathbb{H}}_n^{loc}(t) - \mathbb{Y}_n^{loc}(t) = n^{4/5}\{\widetilde{\mathbb{H}}_n(x_0 + n^{-1/5}t) - \mathbb{Y}_n(x_0 + n^{-1/5}t)\} \geq 0$$

F: Limiting distributions at a fixed point: $k = 2$

Step 3: Preservation of (localized) characterizing relations in the limit

- $\{(\widetilde{\mathbb{H}}_n^{loc}, \widetilde{\mathbb{H}}_n^{loc,(1)}, \widetilde{\mathbb{H}}_n^{loc,(2)}, \widetilde{\mathbb{H}}_n^{loc,(3)})\}_{n\geq 1}$ is tight.

- $\mathbb{Y}_n^{loc} \rightsquigarrow \mathbb{Y}_{a,b}$.

- Fenchel relations satisfied:
  - ▷ $\widetilde{\mathbb{H}}_n^{loc}(t) \geq \mathbb{Y}_n^{loc}(t)$ for all t
  - ▷ $\int_{-\infty}^{\infty}(\widetilde{\mathbb{H}}_n^{loc}(t) - \mathbb{Y}_n^{loc}(t))d\widetilde{\mathbb{H}}_n^{loc,3}(t) = 0$

- Any limit process H for a subsequence $\{\widetilde{\mathbb{H}}_{n'}^{loc}\}$ must satisfy
  - ▷ $H(t) \geq \mathbb{Y}_{a,b}(t)$ for all t.
  - ▷ $\int_{-\infty}^{\infty}(H(t) - \mathbb{Y}_{a,b}(t))dH^{(3)}(t) = 0$.

- Show the process H characterized by these two conditions is unique!

F: Limiting distributions at a fixed point: $k = 2$

Upshot after rescaling to $\mathbb{Y}_{1,1} \equiv \mathbb{Y}$:

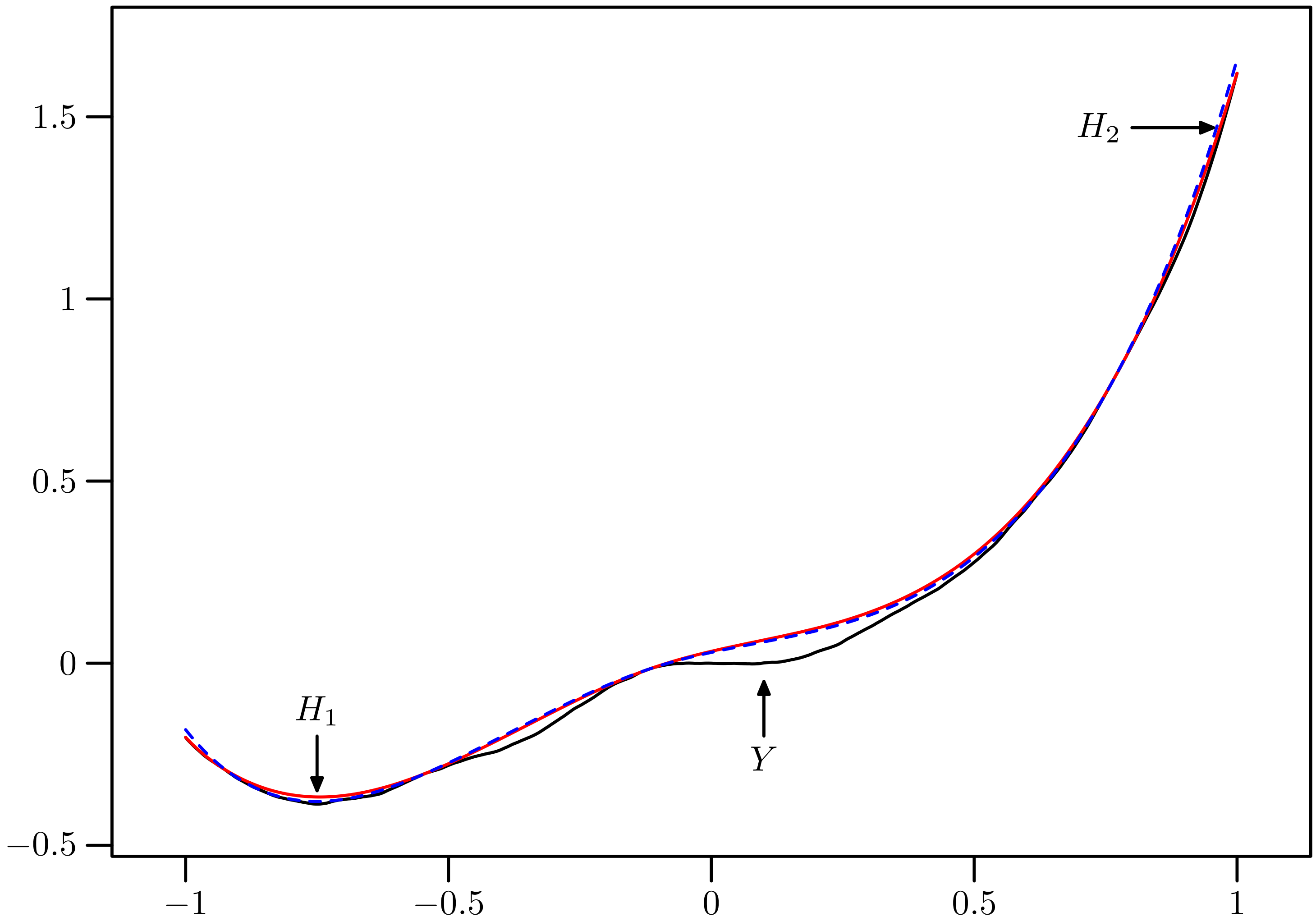
Theorem. (Groeneboom, Jongbloed & W (2001)) If $f \in \mathcal{D}_2$, $f_0(x_0) > 0$, $f_0^{(2)}(x_0) > 0$, and $f_0^{(2)}$ continuous in a neighborhood of x_0, then

$$\begin{pmatrix} n^{2/5}(\widetilde{f}_n(x_0) - f(x_0)) \\ n^{1/5}(\widetilde{f}'_n(x_0) - f'(x_0)) \end{pmatrix} \to_d \begin{pmatrix} c_1(f) H^{(2)}(0) \\ c_2(f) H^{(3)}(0) \end{pmatrix}$$

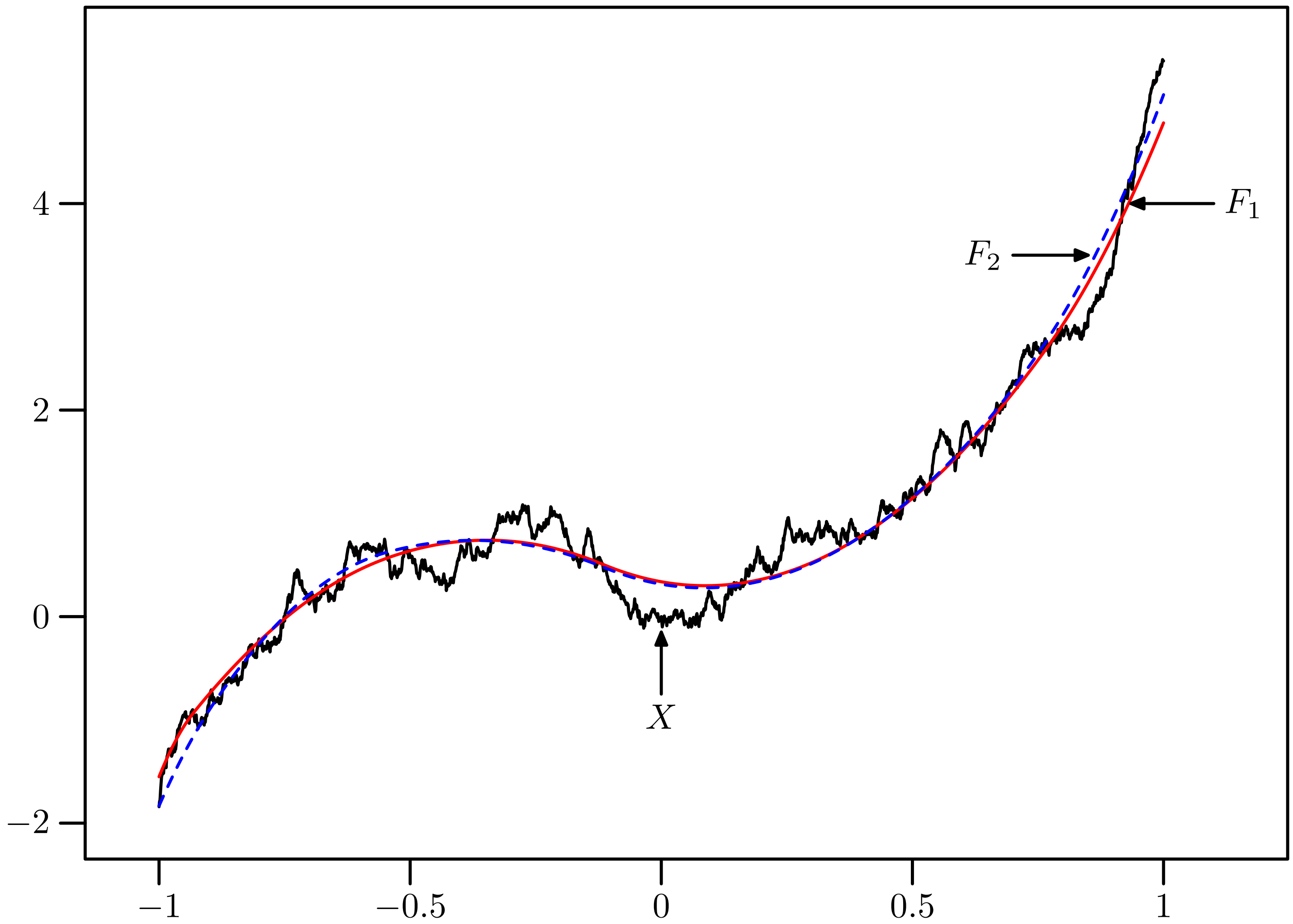
where

$$c_1(f) \equiv \left(\frac{f^2(x_0) f''(x_0)}{4!} \right)^{1/5}, \qquad c_2(f) \equiv \left(\frac{f(x_0) f''(x_0)^3}{4!^3} \right)^{1/5}.$$

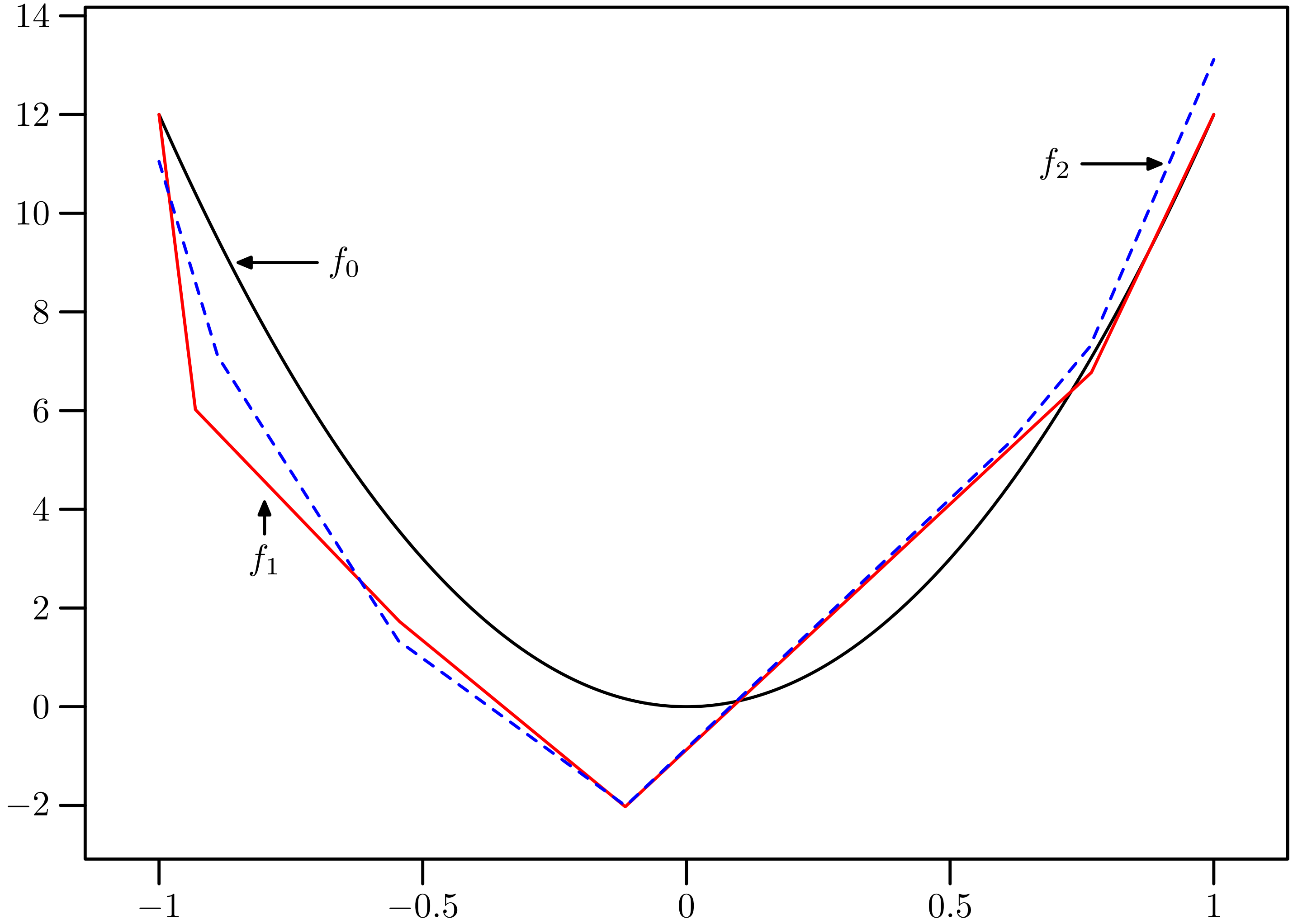
F: Limiting distributions at a fixed point: $k = 2$



F: Limiting distributions at a fixed point: $k = 2$



F: Limiting distributions at a fixed point: $k = 2$



F: Limiting distributions at a fixed point: $k = 2$