The Breslow Estimator, the Breslow-Crowley Theory, and Their Modern Impacts

A Personal History

Jon A. Wellner [1]

[1] Departments of Statistics and Biostatistics
University of Washington

August 4 / In honor of Norman E. Breslow

Outline

1. Introduction: the facts
   - The Breslow estimator
   - The Breslow - Crowley theory
2. The Impacts, Part I
   - Citations
   - Developments
3. The Impacts, Part II

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The Breslow estimator.
of the baseline hazard, Cox model

- $X = (\Delta, T, Z)$,
  $T = \min\{\tilde{T}, C\}$, $\Delta = 1\{\tilde{T} \leq C\}$, $Z =$ vector of covariates.
- $\theta =$ finite-dimensional parameter;
- three infinite - dimensional parameters;
  $\Lambda = \int_0^{\cdot} \lambda(s)ds =$ baseline hazard function,
  $G(t|Z) = P(C \leq t|Z) =$ conditional distribution of censoring time given covariates Z
  G_Z, the marginal distribution of the covariate vector Z.
- $\Lambda(t|Z) = e^{\theta' Z}\Lambda(t) \equiv$ "the Cox model"

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David R. Cox (1975)

Sir David R. Cox (March, 2006)

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  $Y(t) = 1_{[T \ge t]}$ = at risk process.
- Density of $X = (\Delta, T, Z)$ at $x = (\delta, t, z)$ is given by

$$\exp(-e^{\theta' z}\Lambda(t))[e^{\theta' z}\lambda(t)(1 - G(t|z))]^{\delta}[g(t|z)]^{1-\delta} g_Z(z).$$

- Suppose $X_1, \ldots, X_n$ i.i.d. as $X = (\Delta, T, Z)$.
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$$\overline{N}_n(t) = \mathbb{P}_n \Delta 1\{T \le t\} = n^{-1}\sum_{i=1}^{n} N_i(t),$$

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- The Breslow estimator of $\Lambda =$ baseline hazard function is:

$$\widehat{\Lambda}_n(t) = \int_0^t \frac{1}{S_n^0(s;\widehat{\theta})} d\overline{N}_n(s).$$

- where $\widehat{\theta} =$ Cox partial likelikhood estimator of θ.
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Breslow - Crowley theory

limit theory for the Kaplan - Meier estimator

Kaplan-Meier estimator

- With no covariates, let

$$\begin{aligned}\overline{Y}_n(t) &= \mathbb{P}_n Y(t) = \mathbb{P}_n 1\{T \ge t\} \to_{a.s.} P1\{T \ge t\} \\ &= (1 - F(t-))(1 - G(t-)) \equiv 1 - H(t-).\end{aligned}$$

- $\widehat{\Lambda}_n(t) = \int_0^t \frac{1}{\overline{Y}_n(s)} d\overline{N}_n(s)$ estimates $\Lambda(t) = \int_0^t \frac{1}{1-F(s-)} dF(s)$.
- The survival function $1 - F$ is given in terms of Λ by

$$\begin{aligned}1 - F(t) &= \exp(-\Lambda_c(t)) \prod_{s \le t} (1 - \Delta\Lambda(s)) \\ &= \begin{cases} \exp(-\Lambda(t)), & \Lambda \ \text{continuous} \\ \prod_{s \le t}(1 - \Delta\Lambda(s)), & \Lambda \ \text{discrete.}\end{cases}\end{aligned}$$

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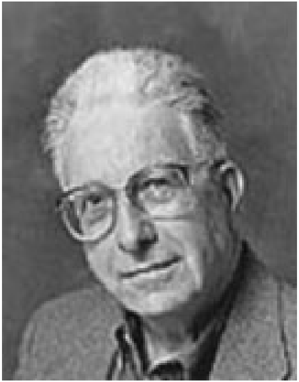
Paul Meier photo



Paul Meier

- Kaplan - Meier (1958) estimator $\widehat{F}_n$ of F is given by

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- Breslow & Crowley (1974) proved:

Theorem

For each $\tau < \inf\{t : H(t) = 1\}$

$$\sqrt{n}(\widehat{F}_n - F) \Rightarrow (1 - F)\mathbb{B}(C) \qquad in \;\; D[0, \tau]$$

where $\mathbb{B}$ is standard Brownian motion on $[0, \infty)$ and

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Impacts & consequences

- Martingale theory:
  Le Cam → Aalen → Gill → Anderson & Gill ...
- Martingale approach to survival analysis: two books, Andersen, Borgan, Gill, and Keiding; Fleming and Harrington
- Dutch school of semiparametrics
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- Empirical process approaches to Breslow-Crowley:
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Odd Aalen

Richard Gill

Per Kragh Andersen

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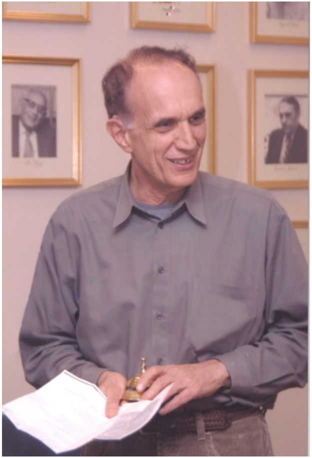

Brad Efron

Impacts - personal I

- John Crowley's (1973) request from Lyon for

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Dick Dudley (1971)

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- 1979 - 1983: efficiency theory for semiparametric models
- one point of origin: trying to understand
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Impacts - personal III

- 1986 - 1996
- getting: out of the office and into the mountains
- three treks in Nepal (1989), (1996), (1999)
- skiing the the Coast Range and Cascades

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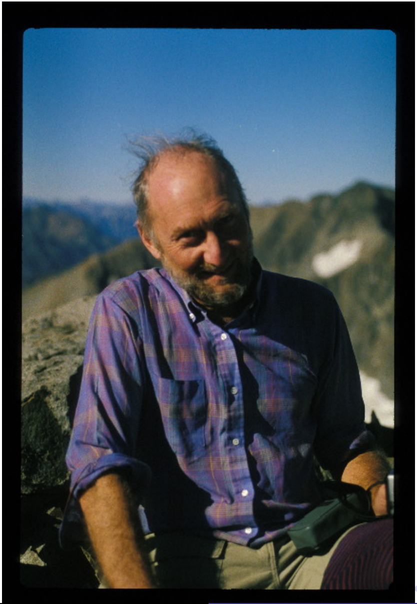

Impacts - personal IV

In his (1972) discussion of D. R. Cox's famous paper on the Cox proportional hazards model, Norman Breslow showed how the baseline cumulative hazard function Λ could be estimated in a natural way in parallel with the estimators of the regression parameters proposed by Cox. The resulting "Breslow estimator", together with Breslow's (1974) joint paper with John Crowley on the asymptotic behavior of the Kaplan-Meier estimator, set the stage for an extraordinary collection of developments in survival analysis, semiparametric models, and counting processes. These developments continue to have significant and profound consequences for present day statistical research and practice.

Impacts - personal IV

Norm has also had a profound effect on my own research directly (and indirectly) over a period of more than 30 years.

THANK YOU NORM