Lecture 2: Some Theory for Estimation with Shape Constraints

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- Talks at YES-I Conference on
  Shape Restricted Inference
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  The Netherlands, October 8-10, 2007
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  http: //www.stat.washington.edu/jaw/jaw.research.html
- Based on joint work with Piet Groeneboom, Geurt Jongbloed;
  former Ph.D. Students Jian Huang, Moulinath Banerjee, Fadoua Balabdaoui, Marloes Maathuis, and Shuguang Song;
  current Ph.D. student Marios Pavlides, current post-doc Hanna Jankowski;
  and the work of many others.

Outline, Lecture 2

- Illustration of the pointwise limit theory pattern:
  convex densities

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- Illustration of the pointwise limit theory pattern:
  log-concave densities
- A functional of interest:
  estimation of the mode

Tentative Outline, Lecture 3

- Illustration of the pointwise limit theory pattern: competing risks with current status data

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- **Step 5.** Weak convergence of the (localized) driving process to a limit (Gaussian) driving process
  empirical process theory: bracketing CLT with functions dependent on n.

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- **Step 8** Cross-check/compare limiting result with local pointwise lower bound theory.
  Use Groeneboom's lower bound lemma (relative of results of Donoho & Liu, Le Cam).

2.1 Illustration: convex decreasing densities

Step 0. $X \sim f$ on $[0, \infty)$ with $f \searrow 0$ and convex ($\mathcal{K}$)

Step 1. Optimization criteron: log-likelihood or least squares

$$\widehat{f}_n = \text{argmax}_{f \in \mathcal{K}} \left\{ \sum_{i=1}^{n} \log f(X_i) \right\}$$

$$\widetilde{f}_n = \text{argmin}_{f \in \mathcal{K}} \psi_n(f)$$

where

$$\psi_n(f) \equiv \frac{1}{2} \int_0^\infty f^2(x)dx - \int_0^\infty f(x) d\mathbb{F}_n(x).$$

Step 2. Characterization: the Fenchel conditions for $\widetilde{f}_n$:
let

$$\widetilde{H}_n(x) \equiv \int_0^x \int_0^y \widetilde{f}_n(t)dtdy \qquad \text{for all } x \in [0,\infty), \quad \text{and}$$

$$\mathbb{Y}_n(x) = \int_0^x \mathbb{F}_n(y)dy$$

Then $\widetilde{f}_n \in \mathcal{K}$ is the LSE if and only if

$$\widetilde{H}_n(x) \geq \mathbb{Y}_n(x) \qquad \text{for all } x > 0,$$

$$\int_0^\infty (\widetilde{H}_n(x) - \mathbb{Y}_n(x))d\widetilde{H}_n^{(3)}(x) = 0,$$

$$\widetilde{H}_n \ \text{ has convex second derivative } \widetilde{f}_n.$$

Step 3. Localization rate / tightness

Proposition. Let x_0 be an interior point of the support of f. For $0 < x \leq y$, define $U_n(x,y)$ by

$$U_n(x,y) \equiv \int_{[x,y]} \{z - (x+y)/2\} d(\mathbb{F}_n - F)(y).$$

Then there exist $\delta > 0$ and $c_0 > 0$ so that, for each $\epsilon > 0$ and x with $|x - x_0| < \delta$,

$$|U_n(x,y)| \leq \epsilon (y-x)^4 + O_p(n^{-4/5}), \qquad 0 \leq y - x_0 \leq c_0.$$

Proposition. Let x_0 and f satisfy $f''(x_0) > 0$ and f'' continuous at x_0. Let $\xi_n \to x_0$, and let

$$\tau_n^- \equiv \max\{t \leq \xi_n : \widetilde{f}_n^{(3)} \text{discontinuous at } t\} \quad \tau_n^+ \equiv \min\{t > \xi_n : \widetilde{f}_n^{(3)} \text{disco}$$

Then $\tau_n^+ - \tau_n^- = O_p(n^{-1/5})$.

Proposition. Suppose $f'(x_0) < 0$, $f''(x_0) > 0$ and f'' continuous in a nbhd. of x_0. Then

$$\sup_{|t| \le M} |\widetilde{f}(x_0 + n^{-1/5}t) - f_0(x_0) - n^{-1/5} t f'(x_0)| = O_p(n^{-2/5}),$$

and

$$\sup_{|t| \le M} |\widetilde{f}'(x_0 + n^{-1/5}t) - f'(x_0)| = O_p(n^{-1/5}).$$

Step 4. Localize the Fenchel relations: define

$$\begin{aligned} \mathbb{Y}_n^{loc}(t) \equiv \; & n^{4/5} \int_{x_0}^{x_0 + n^{-1/5}t} \{ \mathbb{F}_n(v) - \mathbb{F}_n(x_0) \\ & + \int_{x_0}^{v} (f(x_0) + (u - x_0) f(x_0) du \Big\} dv, \end{aligned}$$

$$\begin{aligned}\widetilde{H}_n^{loc}(t) &\equiv n^{4/5}\int_{x_0}^{x_0+n^{-1/5}t}\int_{x_0}^{v}\{\widetilde{f}_n(u)-f(x_0)-(u-x_0)f'(x_0)\}dudv \\ &\quad + \widetilde{A}_n t + \widetilde{B}_n.\end{aligned}$$

Then

$$\widetilde{H}_n^{loc}(t) \ge \mathbb{Y}_n^{loc}(t)$$

with equality if and only if $x_0 + n^{-1/5}t$ is a jump point of $\widetilde{H}_n^{(3)}$.

Note that

$$\begin{aligned}(\widetilde{H}_n^{loc})^{(2)}(t) &= n^{2/5}(\widetilde{f}_n(x_0+n^{-1/5}t)-f(x_0)-n^{-1/5}tf'(x_0)), \\ (\widetilde{H}_n^{loc})^{(3)}(t) &= n^{1/5}(\widetilde{f}_n'(x_0+n^{-1/5}t)-f'(x_0)).\end{aligned}$$

Step 5. Weak convergence of the (localized) driving process $\mathbb{Y}_n$ to a limit (Gaussian) driving process

$$
\begin{aligned}
&\mathbb{Y}_n^{loc}(t) \\
&\quad \stackrel{d}{=} \quad n^{3/10} \int_{x_0}^{x_0+n^{-1/5}t} \{\mathbb{U}_n(F_0(v)) - \mathbb{U}_n(F(x_0))\} dv + \frac{1}{24} f''(x_0) t^4 + o(1) \\
&\quad \Rightarrow \quad \sqrt{f(x_0)} \int_0^t W(s) ds + \frac{1}{24} f''(x_0) t^4
\end{aligned}
$$

by KMT or theorems 2.11.22 or 2.11.23, VdV & W (1996)

$$
\begin{aligned}
&\quad = \quad a \int_0^t W(s) ds + \sigma t^4 \\
&\quad \equiv \quad \mathbb{Y}(t) \equiv \mathbb{Y}_{a,\sigma}(t)
\end{aligned}
$$

where $\mathbb{U}_n(t) \equiv \sqrt{n}(\mathbb{G}_n(t) - t)$ is the empirical process of $\xi_1, \ldots, \xi_n$ i.i.d. Uniform$(0,1)$, $a \equiv \sqrt{f(x_0}$, $\sigma \equiv f''(x_0)/24$.

Step 6. Preservation of (localized) Fenchel relations in the limit.

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  - $H^{(2)}$ is convex.
- Is there a unique such process $H = H_{a,\sigma}$? If so, done!

Step 7. Unique (Gaussian world) estimator resulting from limit Fenchel relations! (Proof: suppose there are two such processes, H_1 and H_2. Then GJW (2001) showed $H_1 = H_2 \equiv H$.)

Upshot: after rescaling to universal ($a = 1$, $\sigma = 1$) limit:

Theorem. If $f \in \mathcal{C}$, $f(x_0) > 0$, $f''(x_0) > 0$, and f'' continuous in a neighborhood of x_0, then

$$\begin{pmatrix} n^{2/5}(\widetilde{f}_n(x_0) - f(x_0)) \\ n^{1/5}(\widetilde{f}'_n(x_0) - f'(x_0)) \end{pmatrix} \to_d \begin{pmatrix} c_1(f) H^{(2)}(0) \\ c_2(f) H^{(3)}(0) \end{pmatrix}$$

where

$$c_1(f) \equiv \left(\frac{f^2(x_0) f''(x_0)}{24} \right)^{1/5}, \qquad c_2(f) \equiv \left(\frac{f(x_0) f''(x_0)^3}{24^3} \right)^{1/5}.$$

Step 8 (or $0'$). Cross-check/compare limiting result with local pointwise lower bound theory.
Use Groeneboom's lower bound lemma (relative of results of Donoho & Liu, Le Cam).

Define f_ϵ by renormalizing (or linearly correcting) $\tilde{f}_\epsilon$ defined by

$$\tilde{f}_\epsilon(x) = \begin{cases} f(x_0 - \epsilon c_\epsilon) + (x - x_0 + \epsilon c_\epsilon) f'(x_0 - \epsilon c_\epsilon), & x \in (x_0 - \epsilon c_\epsilon, x_0 - \epsilon) \\ f(x_0 + \epsilon) + (x - x_0 - \epsilon) f'(x_0 + \epsilon), & x \in (x_0 - \epsilon, x_0 + \epsilon) \\ f(x), & \text{otherwise} \end{cases}$$

where c_ϵ is chosen so that $\tilde{f}_\epsilon$ is continuous at $x_0 - \epsilon$. Let P_n be defined by $f_{\epsilon_n} \equiv f_{\nu n^{-1/5}}$ where

$$\nu \equiv \frac{2 f''(x_0)^2}{5 f(x_0)}.$$

Proposition. If $f(x_0) > 0$, $f''(x_0) > 0$, and f'' is continuous in a neighborhood of x_0,

$$n^{2/5} \inf_{T_n} \max \{E_{n,P_n}|T_n - f_{\epsilon_n}(x_0)|, E_{n,P}|T_n - f(x_0)|\}$$
$$\geq \frac{1}{4}\left(\frac{3}{e\sqrt{2}}\right)^{1/5} \cdot c_1(f),$$

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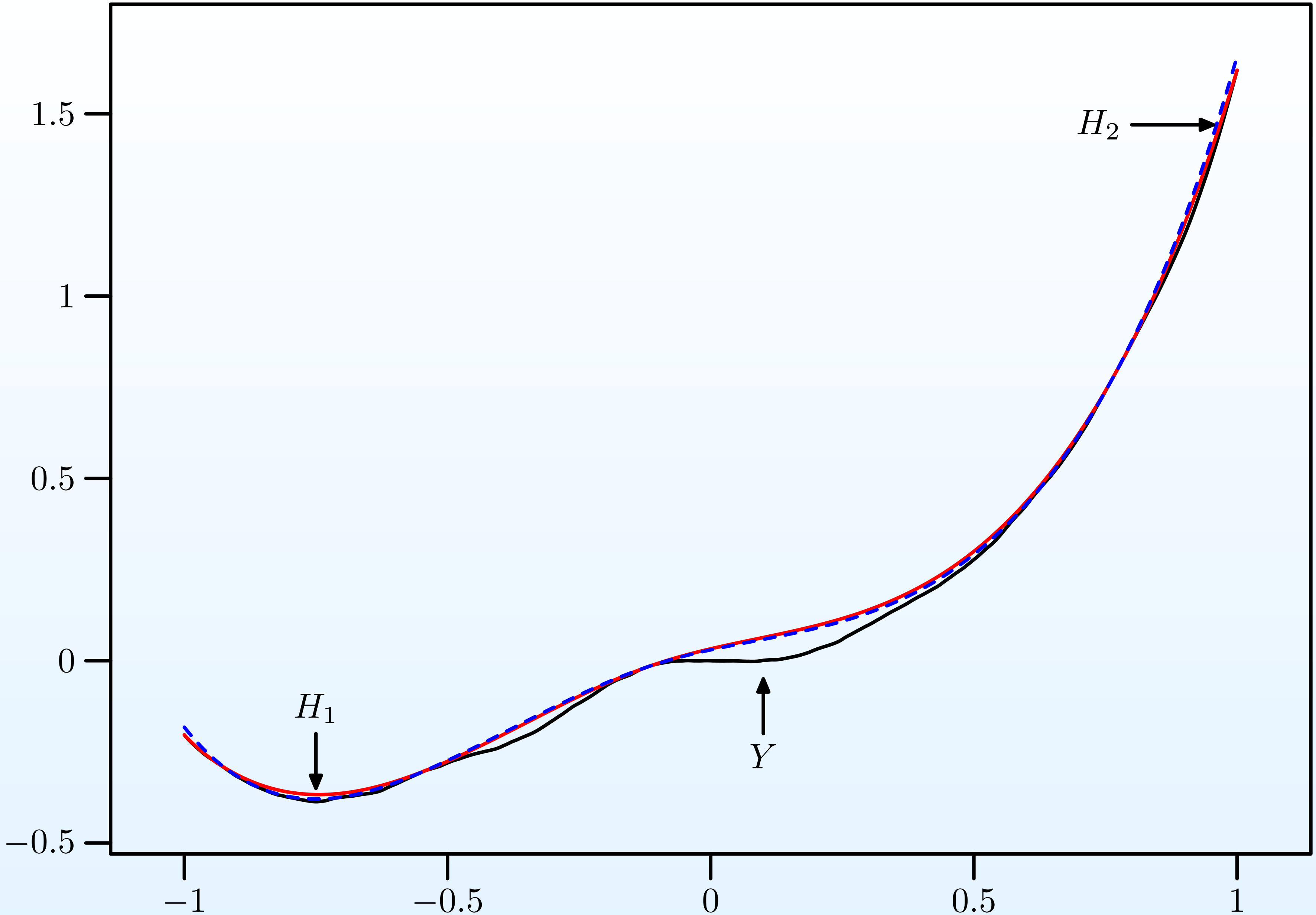
- the “invelope process” H, and the driving process Y
- the derivative process $H^{(1)}$, and the process $Y^{(1)}$
- the concave (limit world estimator of $12t^2$) process $H^{(2)}$
- the piecewise (limit world estimator of $24t$) process $H^{(3)}$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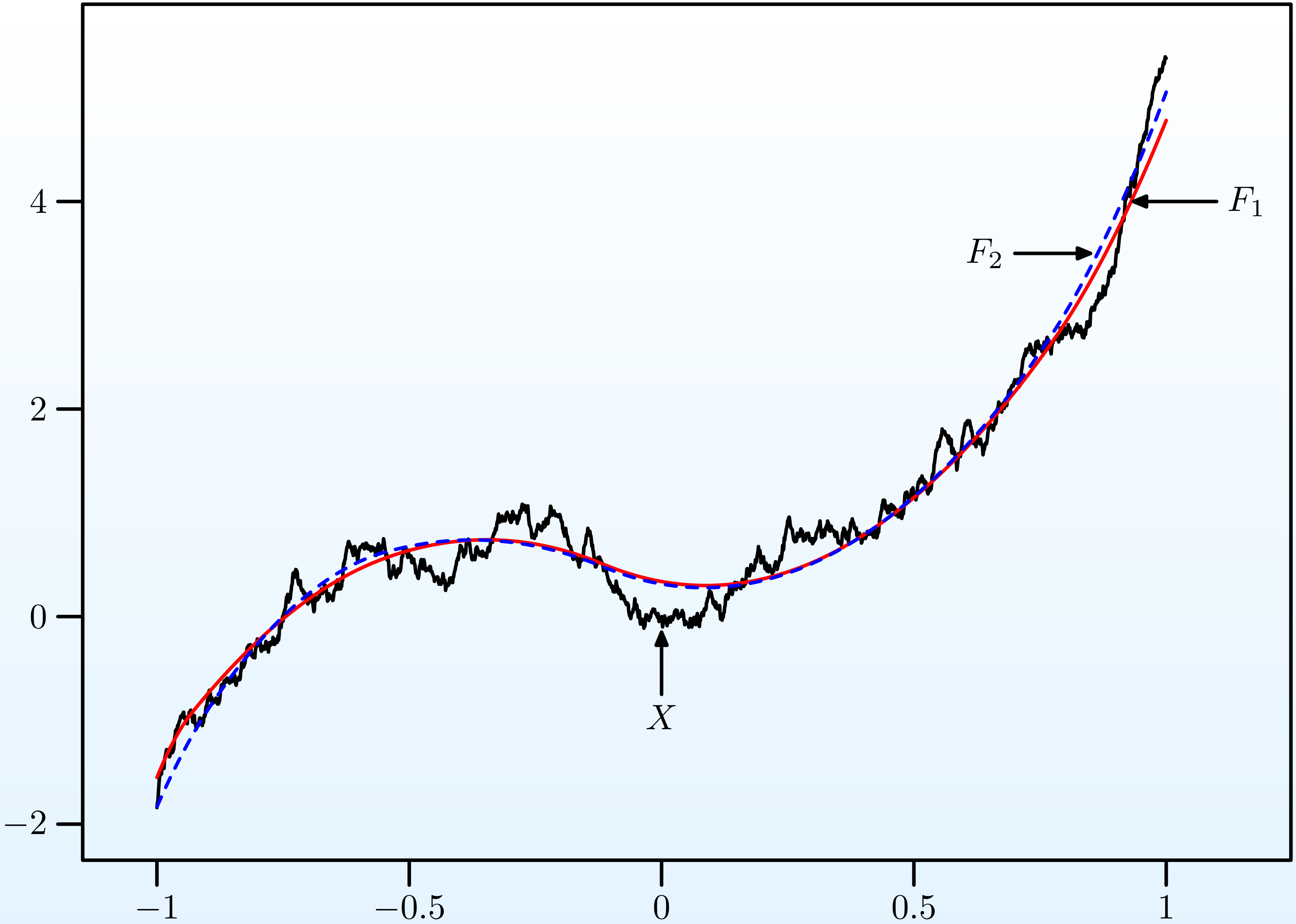
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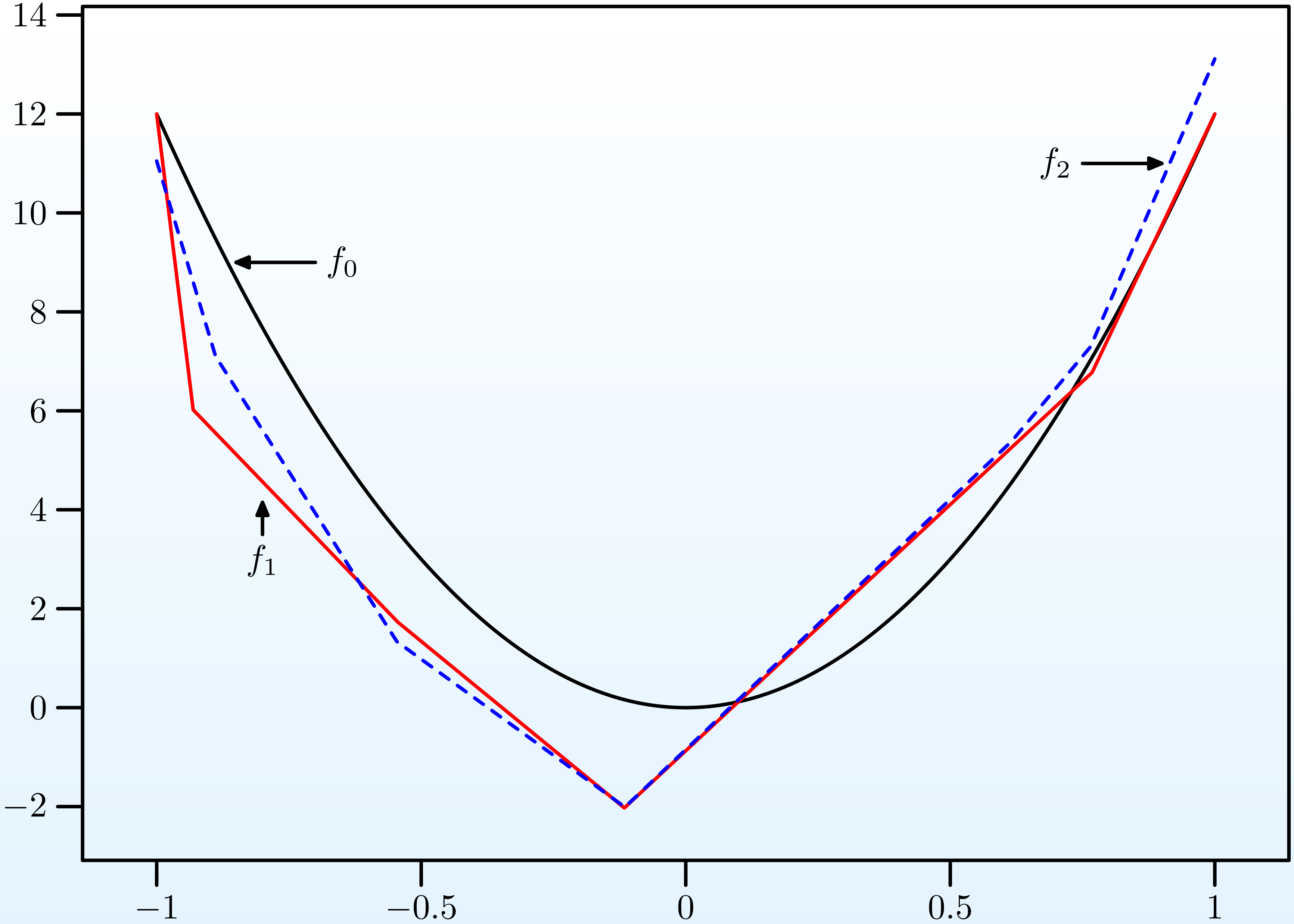

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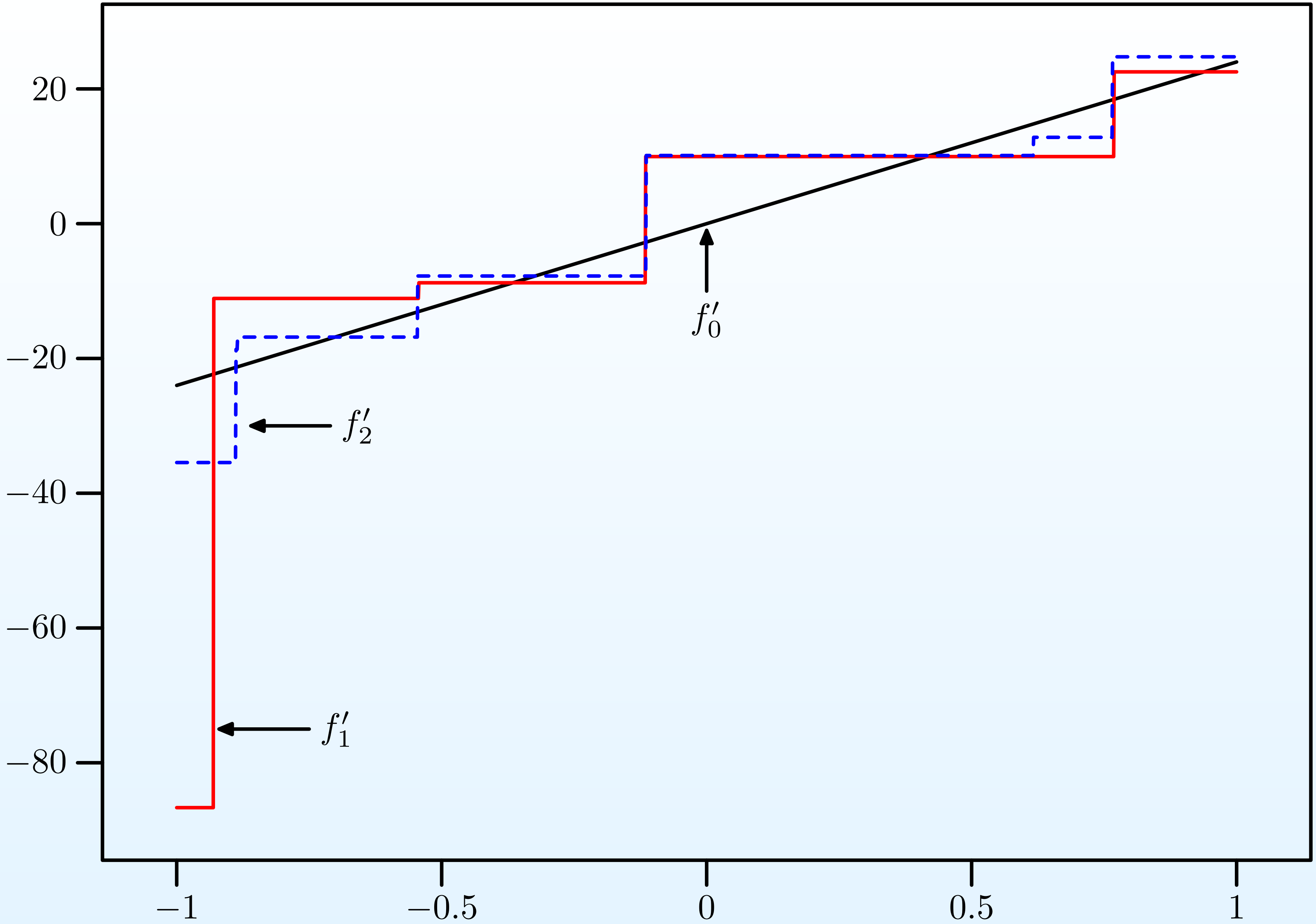
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