Lecture 3: Some Theory for Estimation with Shape Constraints

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  http: //www.stat.washington.edu/jaw/jaw.research.html
- Based on joint work with Piet Groeneboom, Geurt Jongbloed;
  former Ph.D. Students Jian Huang, Moulinath Banerjee, Fadoua Balabdaoui, Marloes Maathuis, and Shuguang Song;
  current Ph.D. student Marios Pavlides, current post-doc Hanna Jankowski;
  and the work of many others.

Outline, Lecture 3

1. Illustration of the pointwise limit theory pattern:
   convex hazards
2. Illustration of the pointwise limit theory pattern:
   log-concave densities
3. A functional of interest:
   estimation of the mode
4. Illustration of the pointwise limit theory pattern:
   competing risks with current status data
5. Partial illustration of the pointwise limit theory pattern:
   $k-$monotone densities
6. Partial illustration of the pointwise limit theory pattern:
   distribution functions and monotone densities on $\mathbb{R}^2$
7. Summary: problems and directions

1. Convex hazards

Nonparametric methods for hazard rate functions.

- Grenander ('56): decreasing case
- Bray, Crawford and Proschan (1967): MLE for U-shaped hazard functions
- Prakasa Rao (1970); Groeneboom (1985); Banerjee (2007): asymptotics
- step-functions
- $n^{-1/3}$ local convergence rates

Here we assume that

$$h(t) \equiv \frac{f(t)}{1 - F(t)} \qquad \text{is convex.}$$

Let $H(t) = \int_0^t h(s)ds$.

Then

$$\mathsf{L}_n(h) = \prod_{i=1}^{n} \left\{ h(X_i) \exp\left[-H(X_i)\right] \right\},$$

and hence the MLE is found by

$$\widehat{h}_n \quad = \quad \text{argmin}_{h \geq 0, \text{convex}} \left\{ \int_0^{\infty} (H - \log h) d\mathbb{F}_n \right\}.$$

Define Empirical CDF and Hazard:

$$\mathbb{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} 1_{[0,t]}(X_i), \quad \mathbb{H}_n(t) = \int_0^t \frac{1}{1-\mathbb{F}_n(s-)} d\mathbb{F}_n(s).$$

Next, fix a $T > 0$. Then the LSE $\widetilde{h}_n$ is defined on $[0, T]$ as

$$\widetilde{h}_n = \text{argmin}_{h\geq 0,\text{convex}} \left\{ \frac{1}{2}\int_0^T h^2(t)dt - \int_0^T h(t)d\mathbb{H}_n(t) \right\}$$

Let $\widetilde{H}_n(t) = \int_0^t \widetilde{h}_n(s)ds$, and $\widetilde{\mathcal{H}}_n(t) = \int_0^t \widetilde{H}_n(s)ds$.
Also, let $\mathbb{Y}_n(t) = \int_0^t \mathbb{H}_n(s)ds$.
The LSE must satisfy:

- $\widetilde{H}(T) = \mathbb{H}_n(T)$ & $\widetilde{\mathcal{H}}_n(T) = \mathbb{Y}_n(T)$
- $\widetilde{\mathcal{H}}_n(t) \geq \mathbb{Y}_n(t)$ for all $t \in [0, T]$
- $\int_0^T (\widetilde{\mathcal{H}}_n - \mathbb{Y}_n)(t) d \left[\widetilde{h}_n\right]'(t) = 0$.

Theorem. Suppose that $h \in \mathcal{K}$ is the true hazard function. Suppose that $h(x_0) > 0$, $h''(x_0) > 0$, and h'' is continuous in a neighborhood of x_0. Then for $\overline{h}_n = \widetilde{h}_n$ or $\overline{h}_n = \widehat{h}_n$

$$\begin{pmatrix} n^{2/5}\{\overline{h}_n(x_0) - h(x_0)\} \\ n^{1/5}\{\overline{h}_n'(x_0) - h'(x_0)\} \end{pmatrix} \rightarrow_d \begin{pmatrix} c_1\, \mathcal{I}^{(2)}(0) \\ c_2\, \mathcal{I}^{(3)}(0) \end{pmatrix}$$

where

$$c_1 = \left(\frac{h^2(x_0) h''(x_0)}{24 S^2(x_0)} \right)^{1/5} \quad \text{and} \quad c_2 = \left(\frac{h(x_0) h''(x_0)^3}{24^3 S(x_0)} \right)^{1/5},$$

for both $\overline{h} = \widetilde{h}_n$ and $\overline{h}_n = \widehat{h}_n$, where $\mathcal{I}$ is the *invelope function* of $\mathbb{Y}(t) \equiv \int_0^t W(s) ds + t^4$: i.e.

- $\mathcal{I}(t) \geq \mathbb{Y}(t)$ for all $t \in \mathbb{R}$.
- $\int_{-\infty}^{\infty} (\mathcal{I}(t) - \mathbb{Y}(t)) d\mathcal{I}^{(3)}(t) = 0$.
- $\mathcal{I}^{(2)}$ is convex.

2. Log-concave densities on $\mathbb{R}$

Suppose that

$$f(x) = \exp(\varphi(x))$$

where φ is concave. The class of all densities f on $\mathbb{R}$ of the form is called the class of *log-concave* densities, $\mathcal{F}_{log-concave}$.

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- Pointwise limit theory? Yes! Balabdaoui and Rufibach (2007)

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  - $\int_{\mathbb{R}} (H_k(t) - Y_k(t)) dH_k^{(3)}(t) = 0$.
  - $H_k^{(2)}$ is concave.

- Pointwise limit theorem for $\widehat{f}_n(x_0)$:

$$\begin{pmatrix} n^{k/(2k+1)}(\widehat{f}_n(x_0) - f(x_0)) \\ n^{(k-1)/(2k+1)}(\widehat{f}'_n(x_0) - f'(x_0)) \end{pmatrix} \to_d \begin{pmatrix} c_k H_k^{(2)}(0) \\ d_k H_k^{(3)}(0) \end{pmatrix}$$

where

$$c_k \equiv \left(\frac{f(x_0)^{k+1}|\varphi^{(k)}(x_0)|}{(k+2)!} \right)^{1/(2k+1)},$$

$$d_k \equiv \left(\frac{f(x_0)^{k+2}|\varphi^{(k)}(x_0)|^3}{[(k+2)!]^3} \right)^{1/(2k+1)}.$$

- Pointwise limit theorem for $\widehat{\varphi}_n(x_0)$:

$$\begin{pmatrix} n^{k/(2k+1)}(\widehat{\varphi}_n(x_0) - \varphi(x_0)) \\ n^{(k-1)/(2k+1)}(\widehat{\varphi}'_n(x_0) - \varphi'(x_0)) \end{pmatrix} \rightarrow_d \begin{pmatrix} C_k H_k^{(2)}(0) \\ D_k H_k^{(3)}(0) \end{pmatrix}$$

where

$$C_k \equiv \left(\frac{|\varphi^{(k)}(x_0)|}{f(x_0)^k (k+2)!} \right)^{1/(2k+1)},$$

$$D_k \equiv \left(\frac{|\varphi^{(k)}(x_0)|^3}{f(x_0)^{k+1} [(k+2)!]^3} \right)^{1/(2k+1)}.$$

3. Estimation of the mode

Let x_m be the *mode* of the log-concave density f, recalling that $\mathcal{F}_{log-concave} \subset \mathcal{F}_{unimodal}$. Lower bound calculations using G. Jongbloed's perturbation of a convex decreasing density, but now perturbing φ yields:

Proposition. If $f \in \mathcal{F}_{log-concave}$ satisfies $f(x_m) > 0$, $f''(x_m) < 0$, and f'' is continuous in a neighborhood of x_m, and T_n is any estimator of the mode $x_m \equiv \nu(P)$, then with P_n corresponding to $f_{\epsilon_n} \equiv \exp(\varphi_{\epsilon_n})$ with $\epsilon_n \equiv \nu n^{-1/5}$ and $\nu \equiv 2f''(x_m)^2/(5f(x_m))$,

$$\liminf_{n\to\infty} n^{1/5} \inf_{T_n} \max \{E_{n,P_n}|T_n - f_{\epsilon_n}(x_m)|, E_{n,P}|T_n - f(x_m)|\}$$

$$\geq \frac{1}{4}\left(\frac{1}{e10}\right)^{1/5}\left(\frac{f(x_m)}{f''(x_m)^2}\right)^{1/5}.$$

On the other hand, the limit theory of Balabdaoui and Rufibach (2007) noted in the previous section implies that the mode estimator derived from the MLE of $\widehat{f}_n$, namely $\widehat{x}_m \equiv \min\{u : \widehat{f}_n(u) = \sup_t \widehat{f}_n(t)\} \equiv M(\widehat{f}_n)$, satisfies

$$n^{1/(2k+1)}(\widehat{x}_m - x_m) \to_d \left(\frac{(4!)^2 f(x_m)}{f''(x_m)^2} \right)^{1/(2k+1)} M(H_k^{(2)})$$

where $M(H_k^{(2)}) = \text{argmax}(H_k^{(2)})$.
Note that when $k = 2$ this agrees with the lower bound calculation, at least up to absolute constants.

4. Competing risks with current status data

See two papers by Groeneboom, Maathuis, and Wellner (2007), *Ann. Statist.* to appear:

`http://www.stat.washington.edu/jaw/jaw_papers.html`

`http://stat.ethz.ch/ maathuis/papers/`

5. $k-$monotone densities

See paper by Balabdaoui and Wellner (2007), *Ann. Statist.*, to appear:

`http://www.stat.washington.edu/jaw/jaw_papers.html`

6. Distribution functions & monotone densities on $\mathbb{R}^2$

Monotone densities on $\mathbb{R}^d$: two types

- Block decreasing: $\mathcal{F}_{BD}$

Consider step 8 in the case of mixtures of uniform monotone densities first first:

6. Distribution functions & monotone densities on $\mathbb{R}^2$

Monotone densities on $\mathbb{R}^d$: two types

- Block decreasing: $\mathcal{F}_{BD}$
- Mixtures of uniform densities (on rectangular densities anchored at 0): $\mathcal{F}_{SMU}$

$$f(\underline{x}) = \int_{(0,\infty)^d} \frac{1}{|\underline{y}|} 1_{(\underline{0},\underline{y}]}(x) dG(\underline{y})$$

for some distribution function G on $(0,\infty)^d$.

Consider step 8 in the case of mixtures of uniform monotone densities first first:

Proposition 1. (Pavlides, 2007). Suppose that $f \in \mathcal{F}_{SMU}$ where $\underline{x}_0 \in (0,\infty)^d$ satisfies $f(\underline{x}_0) > 0$, $\partial f(\underline{x}_0)/\partial x_j < 0$ for $j = 1, \ldots, d$,

$$(-1)^d \frac{\partial^d f(\underline{x})}{\partial x_1 \cdots \partial x_d}\Big|_{\underline{x}=\underline{x}_0} > 0,$$

and the mixed derivative in the last display is continuous on some neighborhood of $\underline{x}_0$. Then there is a sequence $\{f_n\}_{n\ge 1} \subset \mathcal{F}_{SMU}$ such that

$$\liminf_{n\to\infty} \inf_{T_n} \max\left\{E_{f_n} n^{1/3}|T_n - f_n(\underline{x}_0)|, E_f n^{1/3}|T_n - f(\underline{x}_0)|\right\}$$
$$\ge \left(\frac{e^{-1}3^{d-1}}{2^{3d}}\right)^{1/3}\left\{(-1)^d \frac{\partial^d f(\underline{x})}{\partial x_1 \cdots \partial x_d}\Big|_{\underline{x}=\underline{x}_0} \cdot f(\underline{x}_0)\right\}^{1/3}.$$

- Rate of convergence is $n^{1/3}$ for all d.
- Constant reduces to the familiar constant when $d = 1$.
- Shuguang Song (2001): estimation of a distribution function F

with rectangular “current status” censoring.

Proposition 2. (Pavlides, 2007). Suppose that $f \in \mathcal{F}_{BD}$ where $\underline{x}_0 \in (0,\infty)^d$ satisfies $f(\underline{x}_0) > 0$,

$$\frac{\partial f(\underline{x}_0)}{\partial x_j} < 0 \qquad j = 1, \ldots, d,$$

and all the derivatives in the last display are continuous on some neighborhood of $\underline{x}_0$. Then there is a sequence $\{f_n\}_{n\geq 1} \subset \mathcal{F}_{BD}$ such that

$$\liminf_{n\to\infty} \inf_{T_n} \max \left\{ E_{f_n} n^{1/(d+2)} |T_n - f_n(\underline{x}_0)|, E_f n^{1/(d+2)} |T_n - f(\underline{x}_0)| \right\}$$
$$\geq \frac{e^{-1/(d(d+2))}}{4} \left\{ \frac{12 f(\underline{x}_0) d^{d+1}}{2^d (d+2)} \cdot \prod_{j=1}^{d} \left\{ \left| \frac{\partial f(\underline{x})}{\partial x_j} \Big|_{\underline{x}=\underline{x}_0} \right| \right\} \right\}^{1/(d+2)} .$$

Set

$$\widetilde{F}_n(\underline{x}) \equiv \int_{[\underline{0},\underline{x}]} \widetilde{f}_n(\underline{y})d\underline{y}.$$

Then the Fenchel conditions for the LSE $\widetilde{f}_n$ in $\mathcal{F}_{SMU}$ are:

- $\widetilde{F}_n(\underline{x}) \ge \mathbb{F}_n(\underline{x})$ for all $\underline{x}$
- $\int_{(0,\infty)} (\widetilde{F}_n(\underline{x}) - \mathbb{F}_n(\underline{x}) d\widetilde{f}_n(\underline{x}) = 0$

Localization?
Driving process?

Localization of $\mathbb{F}_n$: write $|\underline{t}| \equiv \prod_{j=1}^{d} t_j$.

Let Δ^d denote the $d-$dimensional difference operator:

$$(\Delta^d g)(\underline{u}, \underline{v}) = \sum_{2^d \text{ corners}} (-1)^{\text{par}(v_j)} g(v_j, v_{j+1}).$$

Then I conjecture that

$$\begin{aligned}
\mathbb{Y}_n^{loc}(\underline{t}) &\equiv n^{2/3} \left\{ \Delta^d \mathbb{F}_n(\underline{x}_0, \underline{x}_0 + \underline{t} n^{-1/(3d)}) - \Delta^d F(\underline{x}_0, \underline{x}_0 + \underline{t} n^{-1/(3d)}) \right. \\
&\quad \left. - n^{-1/3} |\underline{t}| f(x_0) \right\} \\
&\Rightarrow \sqrt{f(\underline{x}_0)} W(\underline{t}) - \sigma(f_0) |\underline{t}|^2 \qquad \text{in } C[[-K, K]^d] \\
&\equiv \mathbb{Y}(\underline{t})
\end{aligned}$$

for each $K > 0$ where $W(\underline{t})$ is a 2^d-sided Brownian sheet and

$$\sigma(f_0) \equiv c_d (-1)^d \frac{\partial^d f(\underline{x})}{\partial x_1 \cdots \partial x_d} \Big|_{\underline{x} = \underline{x}_0} \equiv c_d (-1)^d \partial^d f(\underline{x}_0).$$

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n^{1/3}(\widetilde{f}_n(\underline{x}_0) - f(\underline{x}_0)) \quad &\to_d \quad \frac{\partial^d}{\partial t_1 \cdots \partial t_d} \mathbb{H}(\underline{t}) \Big|_{\underline{t}=\underline{0}} \\
&\equiv \quad \partial^d \mathbb{H}(\underline{t}) \Big|_{\underline{t}=\underline{0}}
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  - $\mathbb{H}(\underline{t}) \geq \mathbb{Y}(\underline{t})$
  - $\int_{\mathbb{R}^d} (\mathbb{H}(\underline{t}) - \mathbb{Y}(\underline{t})) d\left\{\partial^d \mathbb{H}(\underline{t})\right\} = 0.$
  - $\Delta^d \{\partial^d \mathbb{H}(\underline{t})\}(\underline{u}, \underline{v}) \geq 0$ for all $\underline{u}, \underline{v} \in \mathbb{R}^d$.

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- Analogues of the results of Groeneboom & Pyke , Wright, Leurgans, Carolan and Dykstra, Anevski and Hössjer for the MLE and LSE of a convex decreasing density? When $f(t) = 2(1-t)_+$, does

$$\sqrt{n}(\widehat{f}_n^{convex}(t) - f(t)) \to \text{Invelope of } \int_0^t \mathbb{U}(F(s))ds?$$

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$$\sqrt{n}(\widehat{f}_n^{convex}(t) - f(t)) \rightarrow \text{Invelope of } \int_0^t \mathbb{U}(F(s))ds?$$

- Banerjee and Wellner (2001) studied likelihood ratio tests of $H : f(t_0) = \theta_0$ versus $K : f(t_0) \neq \theta_0$ in the case of monotone f. Is there a nice theory of pointwise likelihood ratio tests in other shape-constrained problems, e.g. when f is convex?

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  Does the *outline* discussed in my lectures here work?
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  - Pointwise limit theory for log-concave estimators of a density in $\mathbb{R}^2$ or $\mathbb{R}^d$?

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