hw_lect7_1

Consider a Bernoulli dist with parameter pi, (i.e. a population consisting of two types of objects denoted x = 0,1, with the proportion of 1s in the population given by pi). Take samples of size n=3.

a) What's the probability that the minimum of the three numbers is 1?

b) What's the probability that the minimum of the three numbers is 0?

Hint: repeat the derivation of the binomial distribution, i.e. writing out all possibilities, etc. but with X (i.e.,the number of heads out of n) replaced with Min (i.e., the minimum of the three numbers).

possible samples of size 3	X = Min of The 3 numbers	prob
0,0,0	0	$(1-\pi)^3$
0 0 1	0	$(1-\pi)^2\pi$
0 1 0	0	...
1 0 0	0	...
0 1 1	0	$(1-\pi)\pi^2$
1 0 1	0	--
1 1 0	0	--
1 1 1	1	π^3

a) $pr(X=1) = \pi^3$

b) $pr(X=0) =$ 2 ways $= \begin{cases} 1-\pi^3 \\ (1-\pi)^3 + \underbrace{3(1-\pi)^2\pi + 3(1-\pi)\pi^2}_{3(1-\pi)\pi(1-\cancel{\pi}+\cancel{\pi})} \end{cases}$ ← Good enough. But

$$(1-\pi)^3 + 3\pi(1-\pi)$$

Note That you just derived The dist. of minimum of 3 numbers (0/1):

x	0	1
$p(x)$	$1-\pi^3$	π^3

hw_lect7-2
For a period of 10 hours, we observe the number of cars that went through a stop sign (without stopping), per hour. Here is my data: 2, 2, 3, 2, 4, 2, 0, 1, 3, 2. what's the prob that, for a random hour, all cars will stop at the stop sign? Note: What we are given here is data, and so, we can only approximate the distribution parameter(s).

Let X = # of cars that go thru the stop sign, per hour.

Assume X = poisson with param λ.

λ is the parameter of the distribution, ie. something we do not know. But its meaning is the average # of cars that go thru the stop sign, per hour. So, we can approximate it with the average of the 10 counts we have, ie. $\lambda = 2.1$

Then $\text{prob(all will stop)} = \text{pr}(X=0) = \dfrac{e^{-\lambda}\lambda^0}{0!} = e^{-2.1} = \underline{0.12}$

Make sure you realize how amazing this result is! You can actually find the prob. of $X=0$, with almost nothing at all related to $X=0$!

hw-lect 7-3

In The prev. hw problem, you have to approximate The param. of a distr. based on some observed data. In some problems, however, The value of The param. can be obtained from knowledge of The probs Themselves. For example, suppose $x \sim \text{Poiss}(\lambda)$.

a) If $pr(x=0)$ is known, what's The value of λ?

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \Rightarrow pr(x=0) = e^{-\lambda} \Rightarrow \lambda = -\log pr(x=0)$$

b) If The ratio $\frac{pr(x=2)}{pr(x=1)}$ is known, what's The value of λ?

$$\left.\begin{aligned} pr(x=1) &= e^{-\lambda}\lambda \\ pr(x=2) &= \frac{e^{-\lambda}\lambda^2}{2} \end{aligned}\right\} \Rightarrow \frac{pr(x=2)}{pr(x=1)} = \frac{\lambda}{2} \Rightarrow \lambda = 2\,\frac{pr(x=2)}{pr(x=1)}$$

c) Suppose $pr(x=1)$ is known. Then $pr(x=1) = \lambda e^{-\lambda}$.
Plot (by hand) $\lambda e^{-\lambda}$ as a function λ, where $0<\lambda<\infty$.
Clearly mark The maximum value of $\lambda e^{-\lambda}$; call it P.

$$y = \lambda e^{-\lambda}$$

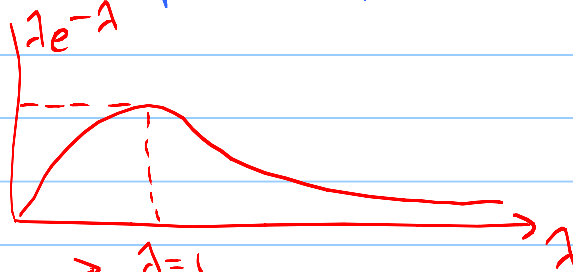
$$y' = e^{-\lambda} - \lambda e^{-\lambda}$$

$$= (1-\lambda)e^{-\lambda}$$

$\therefore\ \lambda = 1 \Rightarrow \max y$

$$y(\lambda=1) = e^{-1}$$

$P = \frac{1}{e}$



d) Finally, suppose in a problem involving some random variable x, whose dist. is not known, we find $pr(x=1) > P$. What does That say about using Poisson dist. to describe x?

Then The poisson dist. cannot be used, because for a variable That does follow poisson, $pr(x=1) < P$.

Big Moral: learn to manipulate The formulas for dists!

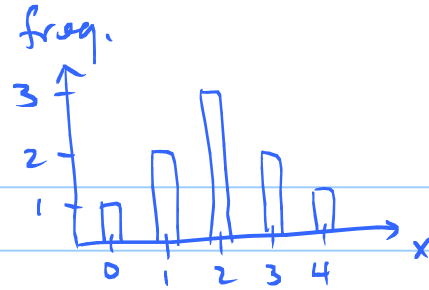
hw_lect8_1

Consider the adjacent histogram.

a) Compute/find the sample mean x_bar. Show work.

b) Find The sample std. dev. s, using the defining formula.

c) Draw the mean and the sample std. dev. on the histogram.



a) $x_i = (0, 1, 1, 2, 2, 2, 3, 3, 4)$

$$\bar{x} = \frac{1}{9}(0+1+1+2+2+2+3+3+4) = \frac{18}{9} = 2$$

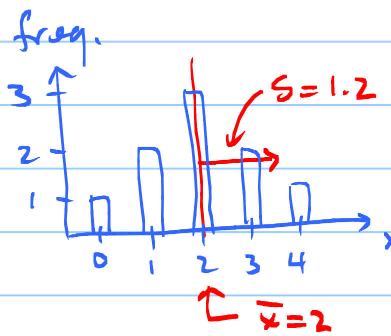
b) $$s^2 = \frac{1}{9-1}\sum_{i=1}^{9}(x_i - \bar{x})^2$$

$$= \frac{1}{8}\Big[(0-2)^2 + (1-2)^2 + (1-2)^2 + (2-2)^2 + (2-2)^2 + (2-2)^2 + (3-2)^2 + (3-2)^2 + (4-2)^2\Big]$$

$$= \frac{1}{8}\left[4 + 1 + 1 + 0 + 0 + 0 + 1 + 1 + 4\right] = \frac{12}{8} = \frac{3}{2} = 1.5$$

$$s = \sqrt{1.5} = 1.22$$

c)

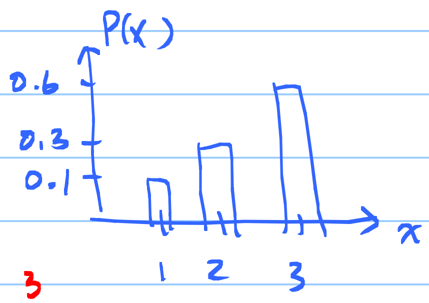


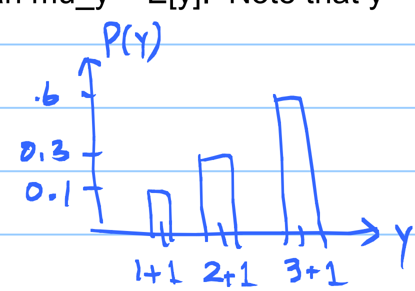
Note That $\bar{x}$ and s agree with The eye-ball estimates I told you about on Day 1.

hw_lect8-2

(a) For the bottom/left distribution, find the distr. mean mu_x = E[x].

(b) For the bottom/right distribution, find the distr. mean mu_y = E[y]. Note that y = x + 1.





a) $$E[x] = \sum_{x=1}^{3} x\,p(x) = 1(.1) + 2(.3) + 3(.6)$$

b) $$E[y] = \sum_{y=2}^{4} y\,p(y) = (1+1)(.1) + (2+1)(.3) + (3+1)(.6)$$

$$= 1(.1) + 2(.3) + 3(.6) + \underbrace{(.1 + .3 + .6)}_{1}$$

$$E[y] = E[x] + 1$$

hw_lect8_3:
To understand our formulas, it is often useful to see what they say if our data are just a bunch of 0's and 1's.
So, suppose our data consists of n0 zeros and n1 ones. Note: n = n0 + n1 .
a) Find the sample mean
b) Show that the sample variance is (n0 * n1)/(n(n-1))

$$\left(\frac{n_0}{n}\right)^2 = \left(\frac{n-n_1}{n}\right)^2$$

a) $\bar{x} = \frac{1}{n}\sum_i^n x_i = \frac{1}{n}\left[n_0(0) + n_1(1)\right] = \frac{n_1}{n}$

b) $s^2 = \frac{1}{n-1}\sum_i (x_i - \bar{x})^2 = \frac{1}{n-1}\left[n_0(0-\bar{x})^2 + n_1(1-\bar{x})^2\right] = \frac{1}{n-1}\left[n_0\left(\frac{n_1}{n}\right)^2 + n_1\left(1-\frac{n_1}{n}\right)^2\right]$

$= \frac{1}{n-1}\frac{n_0 n_1}{n^2}\left[n_1 + n_0\right] = \frac{n_0 n_1}{n(n-1)}$

Altnenatively

$$s^2 = \frac{n}{n-1}\left(\overline{x^2} - \bar{x}^2\right) = \frac{n}{n-1}\left[\frac{n_0(0)^2 + n_1(1)^2}{n} - \left(\frac{n_1}{n}\right)^2\right]$$

$$= \frac{\cancel{n} n_1}{(n-1)\cancel{n}}\left[1 - \frac{n_1}{n}\right] = \frac{n_1}{(n-1)}\frac{n-n_1}{n} = \frac{n_0 n_1}{n(n-1)}$$

Note: if we define $p = \frac{n_1}{n}$, ie. something called The sample proportion,
$\bar{x} = p$, $s^2 = \frac{n}{n-1}p(1-p)$

hw_lect8-4
For the exponential distribution with parameter lambda, find the mean.
Hint: You may use thise integral: $\int_0^\infty y e^{-y}\,dy = 1$

For exponential distr: $f(x) = \lambda e^{-\lambda x}$. then,

$$\mu_x = E[x] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^\infty \lambda x e^{-\lambda x}\,dx$$

$y = \lambda x$
$dy = \lambda dx$

$$= \frac{1}{\lambda}\int_0^\infty y e^{-y}\,dy = \boxed{\frac{1}{\lambda}}.$$