Lecture 7 (Ch 1)

We have played around with lots of distributions. It's time to derive one. Here is The $\boxed{\text{derivation of Binomial}}$:

Consider N objects (population), where
Each object is 1 (Head, Girl, ...) or 0 (Tail, Boy, ...)
Suppose The proportion of 1's in The pop. is known $= \pi$.

population is Bernoulli with param. π

Now, select n (e.g. 3) of The objects (with replacement) = Sample
and note The value of each object.
Repeat many many times (e.g. 10^8)

Q What long-run proportion (of The 10^8) will be 1,1,1? 1,1,0? Etc.

Note: I'm not asking for The prop. of 1's in each Sample.
I'm asking for The prop., out of 10^8 trials, that we get 1,1,1? Etc.

Make sure you understand prop. & prob.

This is The probability of 1,1,1; or what The book calls "long-run prop."

A

		X = # of 1's
pr. of 1,1,1	$= \pi \cdot \pi \cdot \pi$	3
1,1,0	$= \pi \cdot \pi \cdot (1-\pi)$	2
1,0,1	$= \pi (1-\pi) \pi$	2
0,1,1	$= (1-\pi) \pi \pi$	2
Etc.		
0,0,0	$= (1-\pi)(1-\pi)(1-\pi)$	0

$\left.\begin{matrix}2\\2\\2\end{matrix}\right\}$ 3

$$pr(X=3) = 1 \; \pi^3 \qquad \frac{3!}{3!\,(3-3)!}$$

$$pr(X=2) = 3 \; \pi^2 (1-\pi) \qquad \frac{3!}{2!\,(3-2)!}$$

$$pr(X=1) = 3 \; (1-\pi)^2 \pi \qquad \frac{3!}{1!\,(3-1)!}$$

$$pr(X=0) = 1 \; (1-\pi)^3 \qquad \frac{3!}{0!\,(3-0)!}$$

$$pr(X=x) = \frac{3!}{x!\,(3-x)!} \pi^x (1-\pi)^{3-x}$$

$x = 0,1,2,3$

$\therefore$ $pr(X=x) = \frac{n!}{x!\,(n-x)!}\,\pi^x (1-\pi)^{n-x}$ (Table II)

$\| P(X=x)$ $\quad x = 0, 1, 2, \ldots, n \;=\; \#$ of 1's out of n

This is the mass function, $p(x)$, of a binomial variable X.

Because we derived the above expression using proportions, it follows that $\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} prop(x) = 1$.

Some Comments:

$\Rightarrow$ Recall the connection between coin tosses and sampling:

prob. of getting x heads out of n tosses of a coin
or 1 toss of n coins

$\Updownarrow$

The prob. of getting x boys out of a sample of size n.

$\Rightarrow$ What's π? Don't confuse it with $P(X=x)$ [Important!]

For the coin example, it's the prob. of getting a H on one toss.

In the other example, it's the prob of drawing a boy,

i.e. the proportion of boys in the pop.

Note: The prop. of 1's in each sample of size n, does not show up in discussions of Binomial.

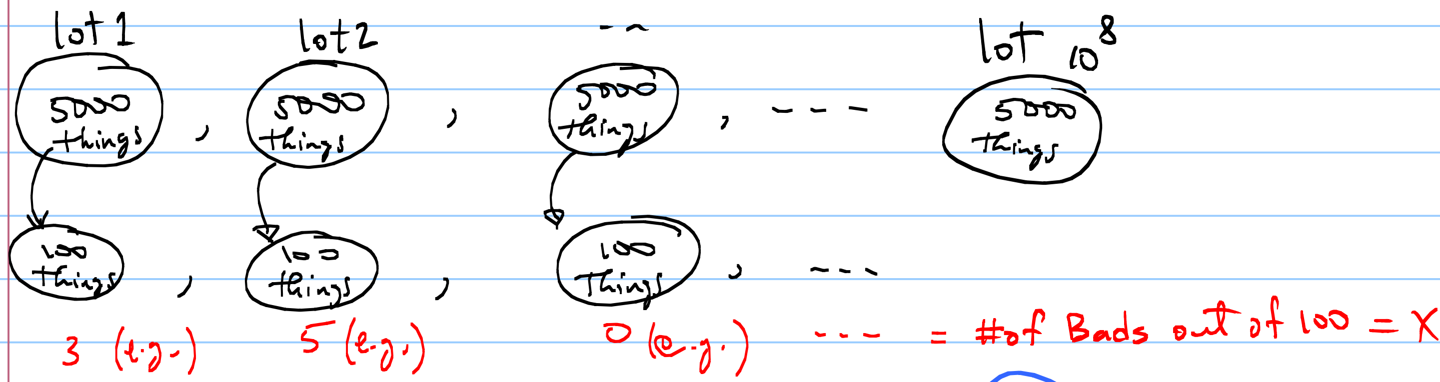
$\Rightarrow$ What's n in Binom(n, π)

A comment about phrases like "take a sample of size x from distribution y."

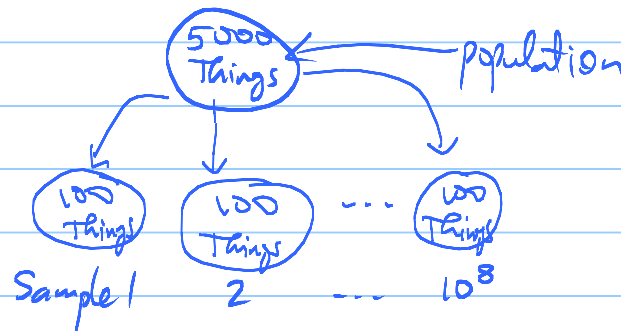
Statements like that are unambiguous when we talk about a distribution like N(mu,sigma). For example, a sample of size 3 from N(mu=1, sigma=2) is 3 (real) numbers, each between -infinity and +infinity, most likely around 1, and with typical spread around 2.

But things get confusing when we talk about "take a sample of size 3 from Binom(n,pi)." In that case, the sample size is NOT the n in Binom(n,pi). Read the last sentence again! The n in Binom(n,pi) *may* be interpreted as a "sample size," but that sample (of size n) would be taken from a Bernoulli(pi), NOT Binomial. This is evident in the derivation of the binomial; there, when we took a "sample of size 3", the sample was taken from a Bernoulli distribution with parameter pi - that's why each sample of size 3 consisted of a sequence of 0s and 1s. By contrast, a sample of size 3 from Binom(n,pi) would a sequence of 3 integers, each between 0 and n. Once again, when we talk about taking a sample of some size from a binomial distribution, the size of the sample is NOT the n parameter of the Binomial.

Example 1.123 (p.56), using binomial.



Assume the lots are identical, i.e. the company manufacturing the 5000 things is extremely consistent. Then, the picture looks like this:



Q What proportion of these 10^8 lots will have $X = 0, 1, ---, 100$?

G = Good
B = Bad

Sample = $\{\underbrace{G, G, ---, G}_{100}\}$

Sample = $\{\underbrace{B, B, ---, B}_{100}\}$

Suppose we know the prop. of Bads in the pop. = 0.5% = .005 = π

Then $P(X = x) = \binom{100}{x} \pi^x (1-\pi)^{100-x}$

prop. of lots with $X = 0$: $\binom{100}{0} \pi^0 (1-\pi)^{100}$ = .6058

$= 1$: $\binom{100}{1} \pi^1 (1-\pi)^{100-1}$ = .3044

$= 2$: = .0757

$= 3$: Etc. = .0124

⋮

Lab.

Important Interpretation

In the long-run we expect
- ~ 60% of the lots to be all good.
- ~ 30% " " " to have 1 bad out of 100.
- ~ 7% " " " 2 bads " " ".
- (i.e. 7% of the lots to be 2% defective)

Now, for large n (a common situation), The $n!$ gets nasty. Also, sometimes π is small. So, consider This limit:

$n \to \infty$, $\pi \to 0$ [ie. rare events] while $n\pi = \text{const.} \equiv \lambda$.

$$\binom{n}{x} \pi^x (1-\pi)^{n-x} \xrightarrow[\text{approx.}]{} \frac{e^{-\lambda} \lambda^x}{x!} = p(x) \text{ of Poisson with param } \lambda.$$

(Table III)

No $n!$, No π. Just λ (average of something, next!)

In The last example, since we know n & π, we can compute λ:

$$\lambda = n \cdot \pi = 100(.005) = 0.5$$

Then $\text{prob.}(X=0) \simeq \frac{e^{-\lambda}\lambda^0}{0!} = e^{-.5} = \boxed{0.6065}$ exact answer $.6058$

Similarly for $\text{prob}(X=1, 2, 3), \ldots$, $\approx$ exact answers from binomial.

Important: Although we derived Poisson(λ) as The limit of Binom(n, π) as $n \to \infty$, $\pi \to 0$ with $\lambda = n \cdot \pi$, it turns out That certain random variables follow The Poisson distribution quite independently of Binomial. Said differently, in some problems we can use Poisson(λ) as long as λ itself is known, in which case we don't even need to know n and π (The Binom params).

In short, if you know n, π, use Binom(n, π).
if you know λ (but not n, π), use Poisson(λ).

But, There is a more fundamental way of knowing which dist to use. Poisson r.v.'s have a certain defining characteristic:

Examples of data that follow the poisson distr:

- \# of bombs dropped over London per block.
- \# of potholes per unit length of roads.
- \# of crashes (cars, planes, buildings) per year.
- \# of people arriving at a website per unit time. $= X$

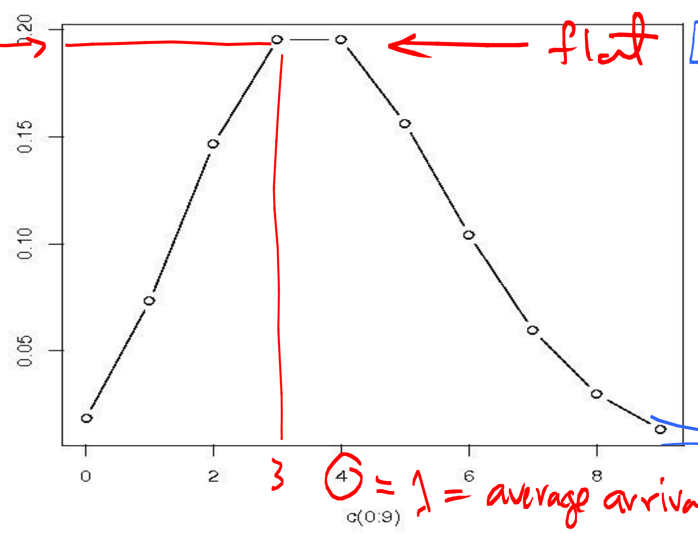
∴ Any r.v. that fits the template "# per unit ..." = Poisson.
And in that case λ = average # per unit ...

Eg. An avg. of 4 people arrive at a website per hour. What's the prob. that 3 people arrive per hour?

Assume X = poisson with $\lambda = 4$ people/hr.

$$P(X=3) = e^{-4}\, 4^3 / 3!$$

$$= 0.19$$



flat [if λ = integer]

Lab: poisson with $\lambda=4$ for $x = 0, 1, \cdots 9$

3   4 = λ = average arrival rate   X

Make sure you are able to recognize the situations that lend themselves to the application of Binomial, Poisson, etc.

hw_lect7_1

Consider a Bernoulli dist with parameter pi, (i.e. a population consisting of two types of objects denoted x = 0,1, with the proportion of 1s in the population given by pi). Take samples of size n=3.

a) What's the probability that the minimum of the three numbers is 1?

b) What's the probability that the minimum of the three numbers is 0?

Hint: repeat the derivation of the binomial distribution, i.e. writing out all possibilities, etc. but with X (i.e., the number of heads out of n) replaced with Min (i.e., the minimum of the three numbers).

hw_lect7-2

For a period of 10 hours, we observe the number of cars that went through a stop sign (without stopping), per hour. Here is my data: 2, 2, 3, 2, 4, 2, 0, 1, 3, 2. what's the prob that, for a random hour, all cars will stop at the stop sign? Note: What we are given here is data, and so, we can only approximate the distribution parameter(s); for now, go ahead and approximate .

hw-lect7-3

In The prev. hw problem, you have to approximate The param. of a distr. based on some observed data. In some problems, however, The value of The param. can be obtained from knowledge of The probs Themselves. For example, suppose $x \sim \text{Poiss}(\lambda)$.

a) If $pr(x=0)$ is known, what's The value of λ?

b) If The ratio $\dfrac{pr(x=2)}{pr(x=1)}$ is known, what's The value of λ?

c) Suppose $pr(x=1)$ is known. Then $pr(x=1) = \lambda e^{-\lambda}$.
Plot (by hand) $\lambda e^{-\lambda}$ as a function λ, where $0<\lambda<\infty$.
Clearly mark The maximum value of $\lambda e^{-\lambda}$; call it P.

d) Finally, suppose in a problem involving some random variable x, whose dist. is not known, we find $pr(x=1) > P$. What does that say about using Poisson dist. to describe x?