STAT 391
Handout 3
The mean of the logistic density

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1 The mean of a symmetric density

Let f be any density of $\mathbb{R}$ that is symmetric around a point x_0 in the sense that $f(x_0 + x) = f(x_0 - x)$ for all x.

We first show that if $X \sim f$ then $E[X] = x_0$.

$$E[X] = \int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{\infty} (x - x_0) f(x)dx + \int_{-\infty}^{\infty} x_0 f(x)dx \quad (1)$$

$$= \underbrace{\int_{-\infty}^{x_0} (x - x_0) f(x)dx}_{\text{chg var } u = x_0 - x} + \underbrace{\int_{x_0}^{\infty} (x - x_0) f(x)dx}_{\text{chg var } u = x - x_0} + x_0 \underbrace{\int_{-\infty}^{\infty} f(x)dx}_{1} \quad (2)$$

$$= \int_{\infty}^{0} -u f(x_0 - u)(-du) + \int_{0}^{\infty} u f(x_0 + u)du + x_0 \quad (3)$$

$$= \int_{0}^{\infty} u \underbrace{(f(x_0 + u) - f(x_0 - u))}_{0} du + x_0 = x_0 \quad (4)$$

Exercise Show that the median of f is also equal to x_0.

2 The logistic density is symmetric around its maximum

We now show that the logistic density given by

$$F(x; a, b) = \frac{1}{1 + e^{-ax-b}}, \quad a > 0 \quad (5)$$

and

$$f(x; a, b) = \frac{a e^{-ax-b}}{(1 + e^{-ax-b})^2} \quad (6)$$

is symmetric around its maximum point $x_0 = -\frac{b}{a}$.

We start with the function $F_0(z) = \frac{1}{1+e^{-z}}$. Note that

$$F_0(z) \;=\; \frac{1}{1+e^{-z}} \;=\; \frac{e^z}{e^z(1+e^{-z})} \;=\; \frac{e^z}{1+e^z} \;=\; 1 - F_0(-z) \tag{7}$$

Now we set $z = ax + b = a(x - x_0)$. Denote by x' the symmetric of x w.r.t. x_0; x' satisfies

$$\frac{x + x'}{2} \;=\; x_0 \tag{8}$$

from which we get $x' = 2x_0 - x$. Let

$$z' \;=\; ax' + b \;=\; a(2x_0 - x) + b \;=\; a(-2\frac{b}{a} - x) + b \;=\; -a(x - x_0) \;=\; -z \tag{9}$$

By replacing the values of z, z' we get that

$$F(x; a, b) \;=\; 1 - F(x'; a, b) \quad \text{whenever } \frac{x + x'}{2} \;=\; -\frac{b}{a} \tag{10}$$

Now we take the derivative of the left and right terms w.r.t x, remembering that $x' = 2x_0 - x$, and we get

$$f(x; a, b) \;=\; -f(x'; a, b)\frac{\partial x'}{\partial x} \;=\; f(x'; a, b) \tag{11}$$

Since the logistic is a symmetric function around $-\frac{b}{a}$, we can conclude that the expectation of the logistic density is

$$E_{f(X;a,b)}[X] \;=\; -\frac{b}{a} \tag{12}$$