STAT 391

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Lecture 1

Logistics

Why — inferences from data

Sample spaces

a science

Probability $\in$ Math

· hypothesis testing

· Predicting natural disasters

Lecture Notes I – (Discrete) Sample spaces and the Multinomial

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Sample space, outcome, event, probability, . . .

Discrete sample spaces

Repeated independent trials, Binomial, Multinomial

Reading: Ch. 2, 3

Basic vocabulary

- S sample space (outcome space)
- $x \in S$ outcome
- $E \subseteq S$ event
- $2^S = \{E \mid E \subseteq S\} \equiv \mathcal{P}(S)$
- $P : 2^S \to [0, 1]$ probability distribution
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Random exp → outcome x

all possible outcomes

1) coin toss $S = \{0, 1\}$ finite

2) die roll $S = \{1, 2, \ldots 6\}$

3) "stopping time" how many times coin tossed until 1 ?

countable

$S = \{1, 01, 001, 0001, \ldots \underbrace{000}_{n-1} 1, \ldots\}$

(discrete)

4) BW images 100 x 100 pixels : $S = 2^{100 \times 100}$

grey level 64 levels, — : $S = 64^{100 \times 100}$

E_1 = contain face ?

E_2 = dark background ?

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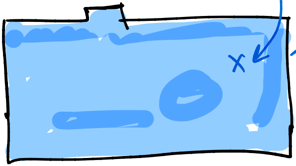
5) random nr. generator rand( )

$S = [0, 1)$

continuous S

(but discrete in reality)

6) Robot location in room



S continuous

map

Event $E \subseteq S$

how many? $2^{|S|}$ when $|S|$ finite

Notation: 2^S = {all subsets of S}

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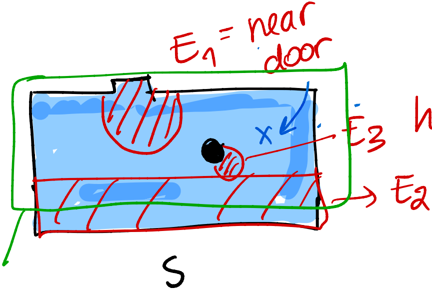
Why events?

- measure theory needs it
- many time we don't care about Prob(x) alone

required by questions sci



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Distribution

$P : 2^S \longrightarrow [0,1]$

$P(E)$ = probability of E

Axioms of Probability

A1. $P(E) \geq 0$

A2. $P(S) = 1$ universe has finite mass

A3. $E \cap E' = \phi$ conservation of mass

$P(E \cup E') = P(E) + P(E')$

(simplified !!)

Random variable

$f : S \rightarrow \mathbb{R}$

$f(x) = y$ deterministic

$x \xmapsto{f} y \in \mathbb{R}$

Ex $x \in S = \{1, \cdots 6\}$

y = parity of x $\in \{0, 1\}$

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Prob $\Longleftrightarrow$ Boolean logic

$A \cup B \Leftrightarrow$ A or B true

$A \cap B \Leftrightarrow$ A and B true

$A \setminus B$

. . . .

Other properties of probability

$E_1, E_2 \quad E_1 \cap E_2 \neq \phi$

P1) $P(\phi) = 0$

A2.3

True

$$\left.\begin{aligned} S \cup \phi &= S \\ S \cap \phi &= \phi \end{aligned}\right\} \Rightarrow$$

$P(\phi) = P(S) - P(S) = 0$

False

P2) $E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2)$

disjoint union

$P(E_1 \setminus E_2) = P(E_1) - P(E_1 \cap E_2)$

$P(E_1 \cup E_2) = P(E_1) - P(E_1 \cap E_2) + P(E_2) - \cancel{P(E_1 \cap E_2)} + \cancel{P(E_1 \cap E_2)}$

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$P(\{x\}) \equiv P(x)$

Notation

Examples

"Uniform" distribution

1) $S = \{0, 1\}$

$P(\{0\}) \equiv P(0) = 1/2$

$P(1) = 1/2$

uniform

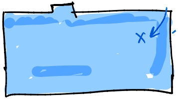
2) $S = \{1, \ldots 6\}$

$P(1) = P(2) = \ldots \quad P(6) = \frac{1}{6}$

3) $S = \{1, 01, 001, \ldots\}$

uniform ← impossible on countable

4)



S

uniform

$P(E) = \dfrac{\text{Area}(E)}{\text{Area}(S)}$

unnormalized prob

normalization const Z

$P(E) \propto \text{Area}(E)$

← proportional

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Physics

$$P(x) \propto e^{-\frac{\text{Energy}(x)}{k_B T}}$$

configuration of system

Discrete sample spaces

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Repeated independent trials, Binomial, Multinomial

A coin is tossed 4 times, and the probability of 1 is $p > 0.5$. The outcomes, their probability and their counts are (in order of decreasing probability):

outcome x	n_0	n_1	$P(x)$	event
1111	0	4	p^4	$E_{0,4}$
1110	1	3	$p^3(1-p)^1$	$E_{1,3}$
1101	1	3	$p^3(1-p)^1$	
1011	1	3	$p^3(1-p)^1$	
0111	1	3	$p^3(1-p)^1$	
1100	2	2	$p^2(1-p)^2$	$E_{2,2}$
1010	2	2	$p^2(1-p)^2$	
1001	2	2	$p^2(1-p)^2$	
0110	2	2	$p^2(1-p)^2$	
0101	2	2	$p^2(1-p)^2$	
0011	2	2	$p^2(1-p)^2$	
0100	3	1	$p^1(1-p)^3$	$E_{3,1}$
1000	3	1	$p^1(1-p)^3$	
0010	3	1	$p^1(1-p)^3$	
0001	3	1	$p^1(1-p)^3$	
0000	4	0	$(1-p)^4$	$E_{4,0}$

Repeated independent trials, Binomial, Multinomial