STAT 391

3/30/23

Lecture 2

Discrete distributions
Repeated independent trials
Normalization
What is statistics all about? VI.

MMP office hour:
Mon 2:30 - 3:30 ↓ 30 min
PDL C-301

TA OH ?

HW1 TB posted 4/5
l1 - mar 28 posted

Lecture Notes I – (Discrete) Sample spaces and the Multinomial

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Sample space, outcome, event, probability,. . .

Normalization

what is stat all about ?

Discrete sample spaces

Repeated independent trials, Binomial, Multinomial

Reading: Ch. 2, 3

Basic vocabulary

- S sample space (outcome space)
- $x \in S$ outcome
- $E \subseteq S$ event
- $2^S = \{E \mid E \subseteq S\} \equiv \mathcal{P}(S)$
- $P : 2^S \to [0,1]$ probability distribution
- $f : S \to \mathbb{R}$ random variable
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$P[0] = 1/11$

$P[1] = 10/11$

Physics:

$$P[x] \propto e^{-\frac{\text{Energy}(x)}{k_B T}}$$

Normalization

Ex: unfair coin

$$P[1] = 10\, P[0]$$

$P[0] \propto 1$ ← unnormalized probabilities

$P[1] \propto 10$

proportional to

$\Leftarrow$ $Z = 1 + 10 = 11$

normalization constant

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Multiple P's on same S

Ex: $S = \{0, 1\}$

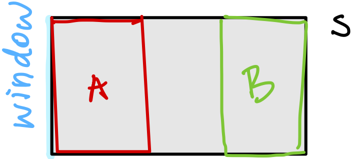
P_0 = uniform $\equiv$ fair coin

P_1 = unfair coin

$P_1[1] = 10/11$, $P_1[0] = 1/11$

P_2 . . .

Robot in room



New info: robot likes light

$\Rightarrow$ new P_1 with $P_1(A) > P_1(B)$

P_0 = uniform

$P_0(A) = \dfrac{\text{Area}(A)}{\text{Area}(S)}$ $\qquad P_0(B)$

Area (B) = Area (A) $\Rightarrow P_0(A) =$

What is STATISTICS all about?

Given: S sample space

unknown $\boxed{P^{true} \text{ on } S}$

data $\mathcal{D} = \{x^1, \dots x^n\}$

* $\mathcal{D} = \{(x^1, y^1), \dots (x^n, y^n)\}$

Model (statistician's job)

distribution P on S

Inference (Reasoning)

Want: Something about outcome (query)

- Ex: $P[\text{event}] \equiv P[x \in E]$
- what is P^{true}?
  Estimation → discrete S
  ↘ continuous S
- prediction
  estimate $P[Y|X]$ *
- hypothesis testing *
  P_1 "better" than P_2?

INFERENCE ⟹ EDUCATED GUESS

- how confident in guess?
- model selection*

Discrete sample spaces

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$S \to$ finite $\{x_1, \ldots x_m\}$ $|S| = m$

$S \to$ countable

$S = \{1, 01, 001, \ldots\}$

$\mathbb{N}^* = \{0, 1, 2, \ldots\}$

$\mathbb{Z} = \{-2, -1, 0, 1, \ldots\}$

S continuous

$S =$ union of intervals in $\mathbb{R}$

$P[A] = \sum_{x \in A} \underbrace{P[x]}_{\theta_x}$ ← if we know θ_x for $x \in S$ we know P

parameters of P

θ_x must satisfy

1) $\theta_x \geq 0$

2) $\sum_{x \in S} \theta_x = 1$

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Ex: unfair die

$\theta_1 = \frac{1}{2} \quad \theta_2 = \theta_3 = \frac{1}{8}$

$\theta_4 = \theta_5 = \theta_6 = \frac{1}{12}$

S countable

$S = \{0, 1, 2, \cdots\}$

Geometric $\quad P[x] \propto \gamma^x$

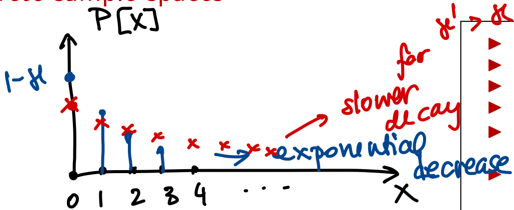
parameter $\gamma \in (0,1)$

$Z = \gamma^0 + \gamma^1 + \gamma^2 + \cdots =$

$= \frac{1}{1-\gamma}$

$P[x] = \underbrace{(1-\gamma)}_{1/Z} \gamma^x$

Discrete sample spaces



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$\boxed{\gamma \leftarrow p}$ in Prob. class

$$P[x] = (1-\gamma)\gamma^x$$

$$x = 0 \Rightarrow P[0] = 1-\gamma$$

$\gamma' > \gamma$

γ controls concentration near 0 or at 0

$\Updownarrow$

γ small $\Rightarrow P[x \gg 0] \approx 0$!!

Discrete sample spaces

Poisson $S = \{0, 1, 2, \dots\}$

$$P[x] = \boxed{\frac{\lambda^x}{x!}} \boxed{e^{-\lambda}} \rightarrow \frac{1}{Z}$$

unnormalized

λ = parameter, $\lambda > 0$

Facts about Poisson

mean $= \lambda$

variance $= \lambda$

stdev $= \sqrt{\lambda}$

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$$\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda} \equiv Z$$

Discrete sample spaces

Repeated independent trials

experiments

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R.V computed from data

$$S \longrightarrow S^{new} = \underbrace{S \times S \times S \cdots \times S}_{\times n}$$

P on S

x outcome

$x^1 \sim P \longrightarrow x^1$ drawn from P

$$(x^1 \; \underbrace{x^2 \; x^3 \cdots x^n}_{\text{drawn independently!}}) = x^{new}$$

(a vector or sequence)

$$P^{new}(x^1, x^2, \ldots x^n) = P[x^1]P[x^2]\cdots P[x^n]$$

$$= \prod_{i=1}^{n} P[x^i]$$

$$= \prod_{j=1}^{m} \theta_j^{n_j}$$

Counts

$n_j \geq 0$ for all $j = 1:m$

$n_1 + n_2 + \ldots n_m = n$ total trials

Statistics

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Discrete sample spaces

Ex $n=4$ die Rolls $S=\{1,\cdots 6\}$

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Ex :

$$\left.\begin{array}{l}\theta_1 = \frac{1}{2} \quad \theta_2 = \theta_3 = \frac{1}{8} \\ \theta_4 = \theta_5 = \theta_6 = \frac{1}{12}\end{array}\right\} P$$

$x = (2, 5, 1, 1)$ $\qquad S^{new} = S^4$

$$P^{new}[x] = P[2]\,P[5]\,P[1]\,P[1]$$

$$= \theta_2 \theta_5 \theta_1^2$$

$$= \theta_1^2 \theta_2^1 \theta_3^0 \theta_4^0 \theta_5^1 \theta_6^0 \rightarrow$$

Counts

$n_1 = 2$

$n_2 = n_5 = 1$

$n_3 = n_4 = n_6 = 0$