# Lecture Notes VI.2 – Double descent on a simple linear regression example

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**Reading**: Ch.10.2.3

# Linear regression when $n < d$

- We describe a very simple linear regression situation (following Bach section 10.2.3)
- For it, we are able to explicitly obtain the expected estimation error $E[\|\theta_{true} - \hat{\theta}\|^2]$
- Surprisingly, the variance of this error decreases with $d$, and the error itself has a limit proportional to $\|\theta_{true}\|^2$.

- Input distribution $x^{1:n} \sim N(0, I_d)$, noise $\epsilon^{1:n} \sim N(0, \sigma^2)$
- Model $y^i = (x^i)^T \theta_{true} + \epsilon^i$.
- Denote $X \in \mathbb{R}^{n \times d}$, $y, \epsilon \in \mathbb{R}^n$ the usual input matrix, output, and noise vectors respectively
- Denote $K = XX^T \in \mathbb{R}^{n \times n}$ the Gram matrix (or kernel matrix). We assume $K$ is non-singular

- From Lecture IV, The Implicit Bias of Gradient Descent we know that
  - When $X$ is full rank $n$, the equation $y = X\theta$ has multiple solutions $\theta$
  - Gradient Descent converges to the min norm solution $\hat{\theta} = X^T K^{-1} y$

# The expected estimation error $MSE(\theta_{true}) = E[\|\theta_{true} - \hat{\theta}\|^2]$

- Decompose $\hat{\theta}$

$$\hat{\theta} \;=\; X^T K^{-1} y \;=\; X^T K^{-1}(X\theta_{true} + \epsilon) \;=\; X^T K^{-1} X \theta_{true} + X^T K^{-1} \epsilon \tag{1}$$

- Then,

$$MSE(\theta_{true}) \;=\; E_{X,\epsilon}[\|\theta_{true} - \hat{\theta}\|^2] \tag{2}$$

$$\;=\; \underbrace{E_X[\theta_{true}^T (I_d - X^T K^{-1} X)\theta_{true}]}_{\text{bias}^2} + \underbrace{E_{X,\epsilon}[\epsilon^T K^{-1} X X^T K^{-1} \epsilon]}_{\text{variance}} \tag{3}$$

- The Variance term becomes

$$Var \;=\; E_{X,\epsilon}[\epsilon^T K^{-1} \epsilon] \tag{4}$$

$$\;=\; E_X[\mathrm{trace}\, K^{-1}]\sigma^2 \qquad \text{Wishart!} \tag{5}$$

$$\;=\; \sigma^2 \frac{n}{d - n - 1} \tag{6}$$

# The expected estimation error $MSE(\theta_{true}) = E[\|\theta_{true} - \hat{\theta}\|^2]$

- The Bias$^2$ term:
- Note that $\theta_P = X^T K^{-1} X \theta_{true}$ is the orthogonal projection of $\theta_{true}$ on the row space of $X$, and $\theta_{true}^T X^T K^{-1} X)\theta_{true} = \|\theta_P\|^2$.
- The subspace is a random subspace of dimention $n$ in $\mathbb{R}^d$. By spherical symmetry, the length of the projection of a fixed vector on a random subspace is the same with that of a projecting a random vector of length (squared) $\|\theta_{true}\|^2$ on a fixed subspace, e.g. the first $d$ unit vectors in $\mathbb{R}^d$. The latter expected value is easy to compute

$$E[\|\theta_P\|^2] \;=\; \frac{n}{d}\|\theta_{true}\|^2 \tag{7}$$

  Exercise Proving this is a moderatly easy exercise
- Hence,

$$\text{bias}^2 \;=\; E_X[\theta_{true}^T(I_d - X^T K^{-1} X)\theta_{true}] \;=\; \frac{d-n}{d}\|\theta_{true}\|^2 \tag{8}$$

# The expected estimation error $MSE(\theta_{true}) = E[\|\theta_{true} - \hat{\theta}\|^2]$

- Finally

$$MSE(\theta_{true}) \;=\; E[\|\theta_{true} - \hat{\theta}\|^2] \;=\; \frac{d-n}{d}\|\theta_{true}\|^2 + \sigma^2 \frac{n}{d-n-1} \tag{9}$$

  for $d > n+1$
- When $d \to \infty$, the variance$\to 0$ and the bias$^2 \to \|\theta_{true}\|^2$