STAT 391
Final Exam
10:30 – 12:20 on June 10, 2010

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Student Name:____

You are allowed 8 pages of notes and the textbook

No electronic devices of any kind are allowed during the exam.

Any fact that was proved in the lectures or in the notes can be used without proof.

Prob.1 ____of 9

Prob.2 ____of 3

Prob.3 ____of 5

Prob.4 ____of 21

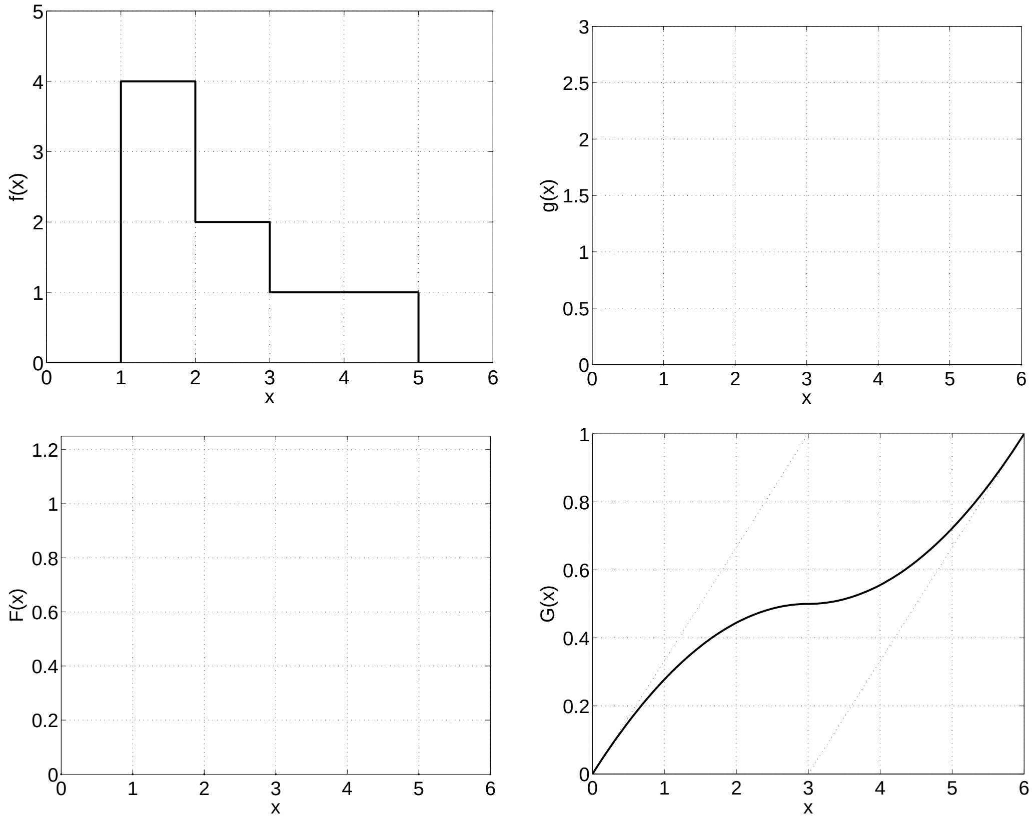
Prob.5 ____of 4

Prob.6 ____of 18

TOTAL: ______of 60

Problem 1 (9 points)

$Z_f =$ $Z_g =$



1.1 Compute the normalization constant Z_f for the density f in the graph on the left. 1 p.

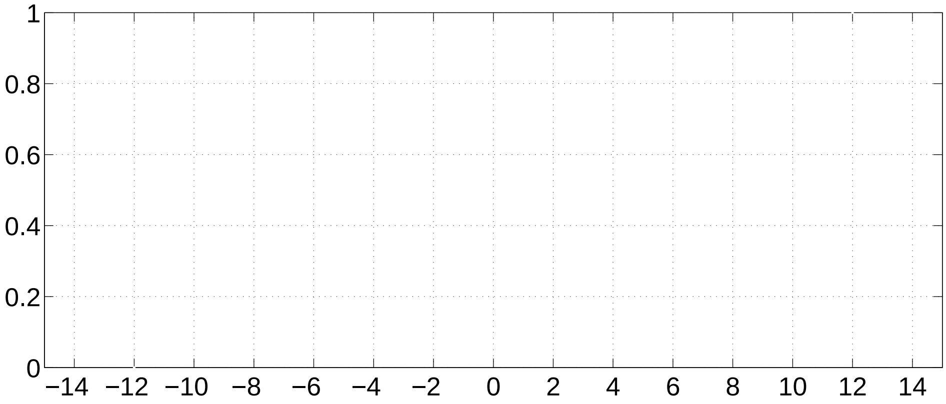
1.2 Make a plot of the CDF $F(x)$ that corresponds to the density f in the graph on the left. (Be as precise as possible). 2 p.

1.3 Make a plot of the density $g(x)$ that corresponds to the CDF G in the graph on the right. (Be as precise as possible). You can choose to draw an unnormalized density; in this case enter the normalization constant Z_g in the space above the graph. 3 p.

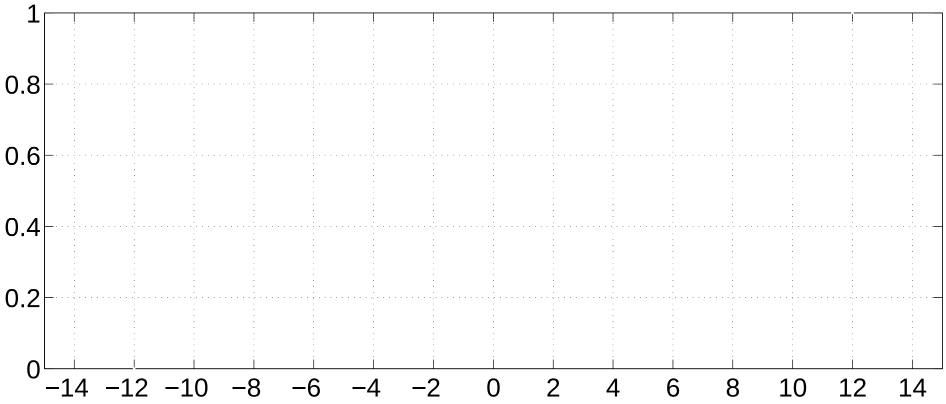
1.4 Make a plot of the density and CDF of the Normal distribution $N(-4, 4)$. Mark the locations of μ, $\mu + \sigma, \mu - \sigma$ on the X-axis of each plot. 3 p.

FYI: $1/\sqrt{2\pi} \approx 0.4$, $\sqrt{2\pi} \approx 2.5$

$Normal(-4, 4)$ density



$Normal(-4, 4)$ CDF



Problem 2 (3 points)

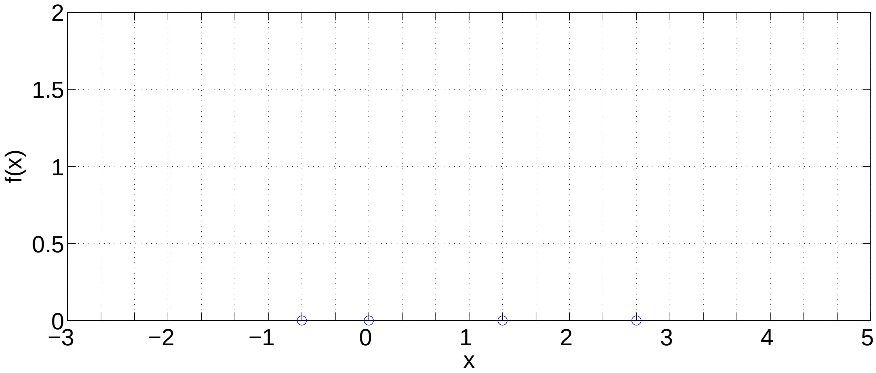
The figure depicts a data set of $n = 4$ points, $\mathcal{D} = \{-2/3, 0, 4/3, 8/3\}$, on the real axis. Draw a kernel density estimate $f(x)$ based on this data set, using a **square kernel** with $h = 2$.

You can choose to draw an unnormalized density; in this case write value of the normalization constant Z above the graph. If your density is normalized, write $Z = 1$ above the graph.

The square kernel is described by

$$k(z) = \begin{cases} 1 & \text{if } z \in [-0.5, 0.5] \\ 0 & \text{otherwise} \end{cases}$$

$Z =$



Problem 3. For this problem, support all your answers and show all your work. (5 points)

The Laplace density is defined on $S = (-\infty, \infty)$ by

$$f(x) = \frac{1}{Z} e^{-\lambda |x|}$$

where $|x|$ denotes the magnitude of x, and $\lambda > 0$ is a parameter.

3.1 Calculate the value of the normalization constant Z as a function of the parameter λ. 1 p.

3.2 We observe the data $\mathcal{D} = \{x_1, x_2, \ldots x_n\}$. 2 p.

Write the expression of the log-likelihood for this data set.

3.3 Find the expression of the Maximum Likelihood estimate of λ based on the log-likelihood derived above. 2 p.

Calculate the numerical value of λ^{ML} for the dataset $\mathcal{D} = \{1, -4, 0, -3, 2\}$.

Problem 4 (21 points)

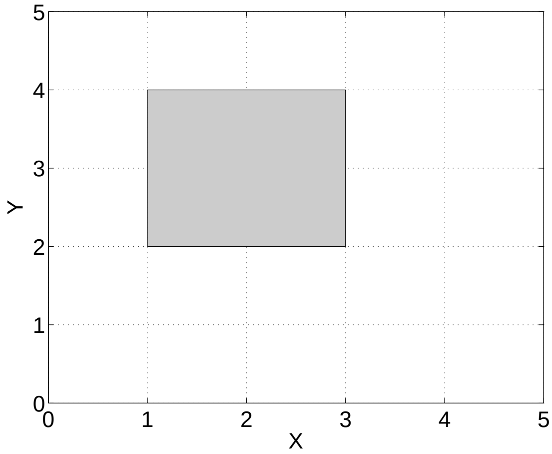
The following are bivariate densities over $(-\infty, \infty)$. The first three are unnormalized uniform densities which take value 1 over the filled in domain. The fourth one is Gaussian, and points from the distribution are shown to indicate the density.

4.1 Enter the normalization constant Z corresponding to the densities in **A, B** above their graphs. 2 p.

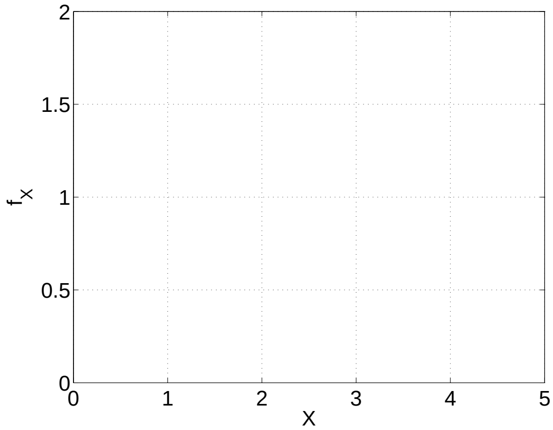
4.2 Plot f_X the marginal density of X below the respective joint densities. 5 p.

A Uniform

$Z =$

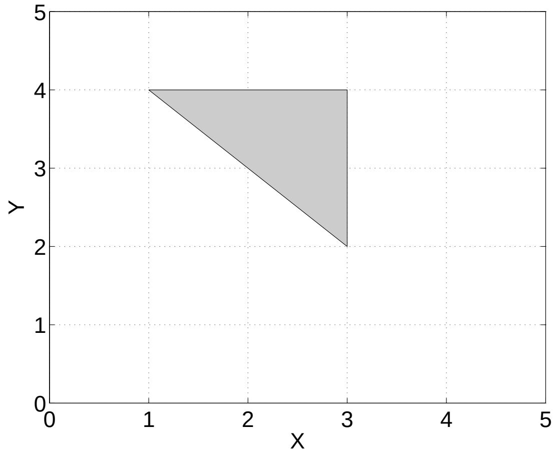


$f_X(x)$

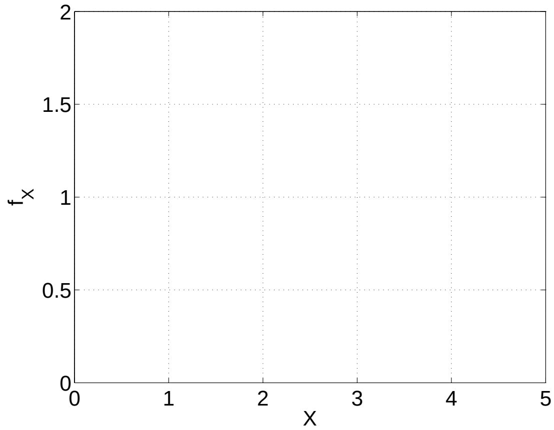


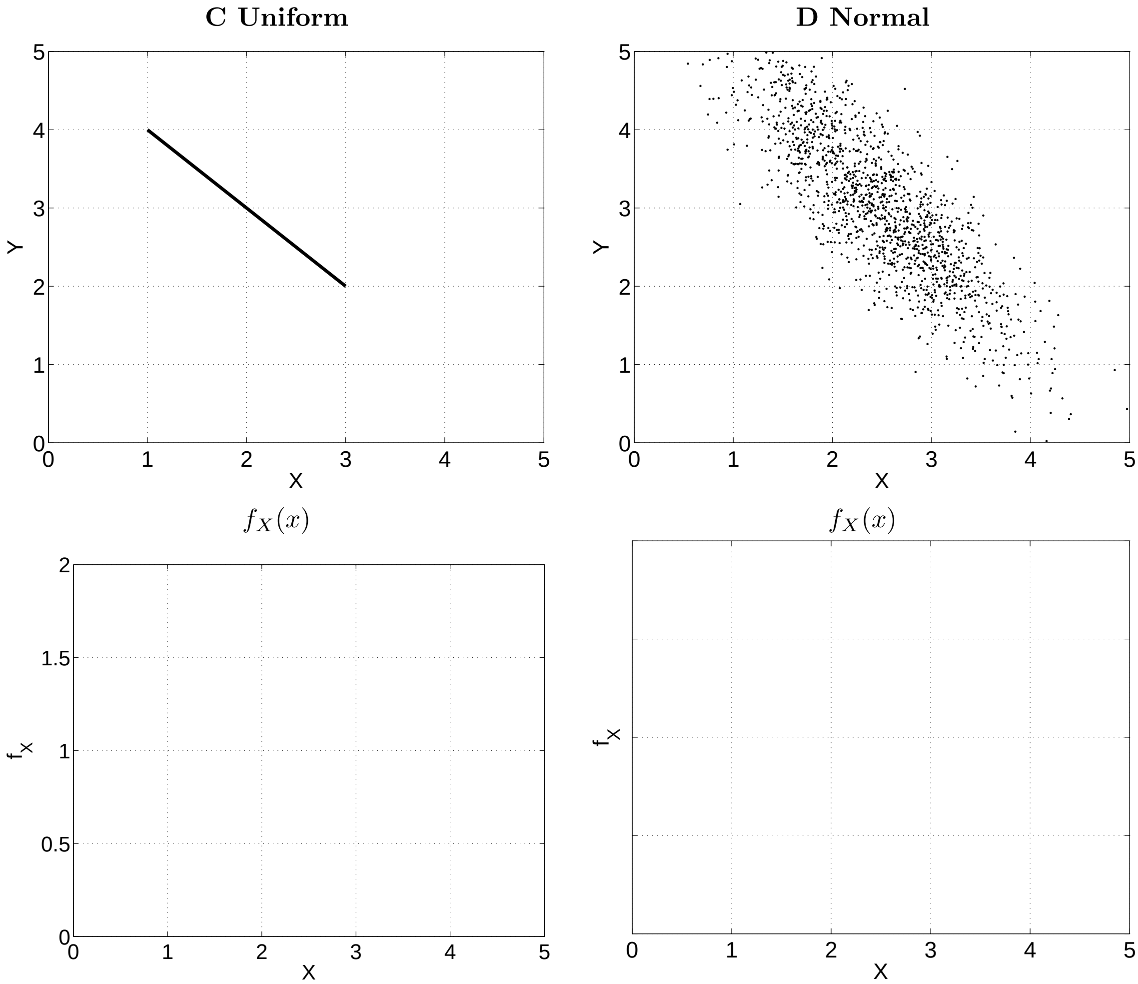
B Uniform

$Z =$



$f_X(x)$




C Uniform
D Normal
Y
X
$f_X(x)$
f_X
0
1
2
3
4
5
0.5
1.5

4.3 Are X and Y independent? 4 p.

A: ☐**Yes** ☐**No** **B:** ☐**Yes** ☐**No**
C: ☐**Yes** ☐**No** **D:** ☐**Yes** ☐**No**

4.4 Mark the location of the point $(E[X], E[Y])$ on each of the graphs. 5 p.

4.5 The correlation coefficient ρ_{XY} is 3 p.

A	**C**	**D**
☐$\rho_{XY} = 0$	☐$\rho_{XY} = 0$	☐$\rho_{XY} = 0$
☐$\rho_{XY} > 0$	☐$\rho_{XY} > 0$	☐$\rho_{XY} > 0$
☐$\rho_{XY} < 0$	☐$\rho_{XY} < 0$	☐$\rho_{XY} < 0$

4.6 What is $P[Y \leq X]$ for **A, B**. 2 p.

A
$P[Y \leq X] =$

B
$P[Y \leq X] =$

Problem 5 (4 points)

For this problem, support all your answers and show all your work.

5.1 Prove that for any two random variables X, Y 2 p.

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

5.2 For two random variables X, Y, we denote $Cov(X,Y) = c$. Calculate $Cov(2X, 3Y)$ as a function of c. 2 p.

Problem 6 (18 points)

For this problem, support all your answers and show all your work.

The CSE building has an intelligent elevator, named Ella, who carries visitors to the 6 floors of the building and to the basement (floor 0). In her idle time, Ella exercises her mind with probability questions. Can you help her find the answers?

Assume in the following that even if a person goes to floor 1, they will still stop by the elevator. Also assume that people entering the building request a floor independently of the floors chosen by other people.

The probability θ_j that a given person entering the building goes to floor j is given in the table below, which also shows the locations of some research groups in the building.

Probability θ_j	**Floor**	**Groups**
$\theta_6 = 6/40$	**6**	G S
$\theta_5 = 8/40$	**5**	T AI S CB
$\theta_4 = 5/40$	**4**	
$\theta_3 = 6/40$	**3**	AI S
$\theta_2 = 5/40$	**2**	
$\theta_1 = 7/40$	**1**	
$\theta_0 = 3/40$	**0**	G

Use these probabilities to answer the following questions.

6.1 What is the probability that a person goes to a floor higher than 3? 1 p.

6.2 In front of the elevator are 4 persons. What is the probability that at least one of them is going to a floor higher than 3? (Literal answer only) 1 p.

6.3 The Artificial Intelligence (AI), Computational Biology (CB), Graphics (G), Systems (S) and Theory (T) groups are located according to the "map" above. In other words, AI groups are on the 3-rd and 5-th floors, graphics on 0 and 6, etc. Assume that there are no other groups on floors 0, 3, 5 and 6, that a visitor to floor j will be going to only one group, and that groups on the same floor have equal probabilities of being visited. E.g a visitor to floor 6 will go to Graphics w.p 0.5 and to Systems w.p 0.5. 4 p.

Calculate the probability that a person goes each of the AI, G, S and T and CB groups.

6.4 What is the probability that a person goes to the 5th floor, given that they are headed for the Theory or the Systems group? 2 p.

6.6 Today is a holiday, but it is also a deadline for a major AI conference. Therefore, a person going to the AI group will stay more than 8 hours (call this time "long") with probability $p = 0.7$, while a person going to one of the other groups will stay long with probability $q = 0.2$. The duration of staying depends only on the group the person is going to and is independent of anything else. 10 p.

Alice and Bob went to the same group, which is one of the aforementioned groups (call this event E_1). Alice got off before the 4-th floor (call this event E_A) and Bob stayed long (call this event E_B). Under these conditions, what is the probability that they went to the AI group?

[extra space on next page]

Have a nice summer!