

.....

STAT 391 Midterm Exam
Thursday May 8 2014, 11:30-12:20

Student name:

- 3 pages of notes allowed
- no other sources of information are allowed
- electronic devices are not allowed

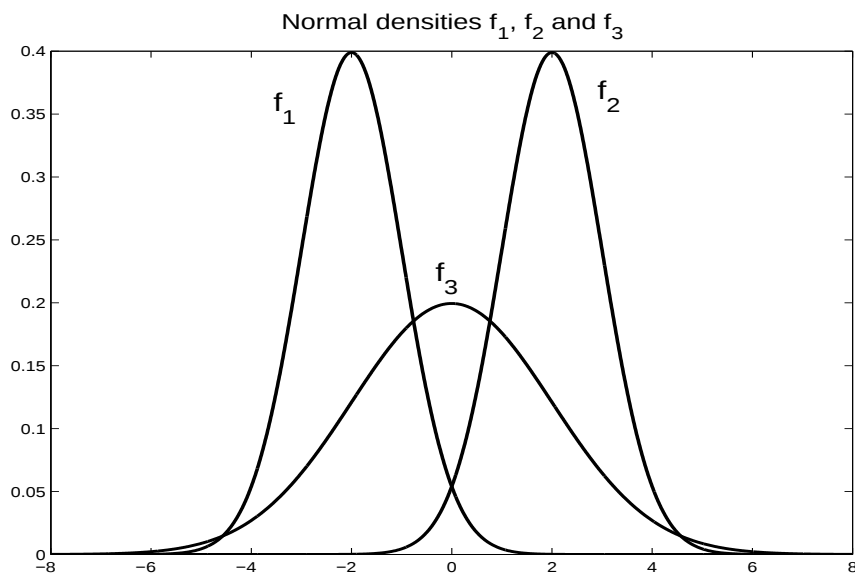
- *Do Well!*

Problem 1	4 points	(Normal distributions)
Problem 2	2 points	(Discrete probabilities)
Problem 3	4 points	(Properties of Expectation and Variance)
Problem 4	8.5 points	(ML estimation with missing information)
Bonus	0.5 points	
Total	24 points	
Extra credit	3 points	

5 points

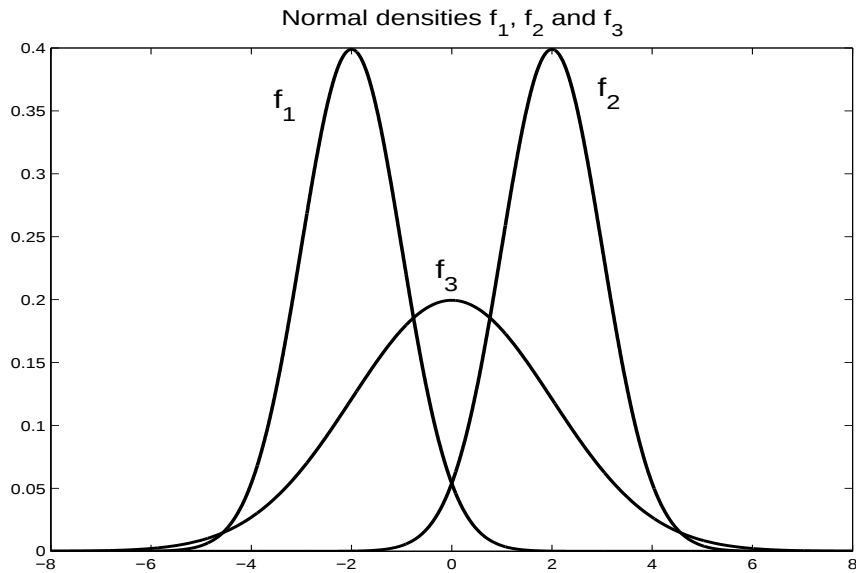
Problem 1 – Normal distribution

The graph below shows three normal densities f_1, f_2, f_3 having parameters $(\mu_1, \sigma_1), (\mu_2, \sigma_2), (\mu_3, \sigma_3)$, and denote by P_1, P_2 and P_3 the probability distributions associated with f_1, f_2 and f_3 , respectively.



- a. Which density has the largest mean μ ?
- b. Which density has the largest standard deviation σ ?
- c. Mark on the graph shown above the positions of μ_1, μ_2, μ_3 .

d. Let A denote the event $x \geq \mu_3$. Draw A on the x axis of the figure below (this is the same figure as on the previous page).



e. $P_3(A)$ the probability of event A under the distribution represented by f_3 is (choose one):

☐	0
☐	1

☐	close to 0
☐	close to 1

☐	1/2
--------------------------	-----

☐	close to 1/2
--------------------------	--------------

f. $P_1(A)$ the probability of event A under the distribution represented by f_1 is (choose one):

☐	0
☐	1

☐	close to 0
☐	close to 1

☐	1/2
--------------------------	-----

☐	close to 1/2
--------------------------	--------------

g. $P_2(A)$ the probability of event A under the distribution represented by f_2 is (choose one):

☐	0
☐	1

☐	close to 0
☐	close to 1

☐	1/2
--------------------------	-----

☐	close to 1/2
--------------------------	--------------

h. Is the statement " $P_1(A) = 1 - P_2(A)$ " true or false?

Problem 2 – Discrete distributionsShow your work

Let $S = \{-1, 0, 1\}$ be the outcome space of an experiment, and P defined by $(\theta_-, \theta_0, \theta_+)$ a distribution over S .

a. Give answers to the following questions as functions of $(\theta_-, \theta_0, \theta_+)$.

$$P(X < 1) =$$

(X_1, X_2, X_3) are three independent samples from P . What is the probability that they are all different?

(X_1, X_2, X_3) are three independent samples from P . What is the probability that $X_1 = X_2 = X_3 = 0$?

(X_1, X_2, X_3) are three independent samples from P . What is the probability that $X_1 = X_2 = X_3$?

b. Assume $\theta_0 = \frac{1}{2}$, $\theta_- = \frac{1}{4}$, $\theta_+ = \frac{1}{4}$.
Calculate $E[X]$.

Calculate $E[X^2]$.

c. The following dataset of $n = 10$ points was sampled i.i.d. from P . $\mathcal{D} = \{-1, -1, -1, 1, 1, 0, 0, 0, 0, 0, \}$. What are the Maximum Likelihood estimates of $(\theta_-, \theta_0, \theta_+)$ from this data set? *No need to show work. Numerical result sufficient.*

[**d. – Extra credit**] Assume now that instead of P , there is another distribution $\tilde{P}$ on S from which $\mathcal{D}$ was sampled. $\tilde{P}$ has parameters $(\tilde{\theta}_-, \tilde{\theta}_0, \tilde{\theta}_+)$ with the property that $\tilde{\theta}_- = \tilde{\theta}_+ = \tilde{\theta}_1$. (In other words, the parameter of the model $\tilde{P}$ are *tied*, or *have constraints*.) Write the likelihood of $(\tilde{\theta}_0, \tilde{\theta}_1)$ given the dataset $\mathcal{D}$.

[**e. – Extra credit**] Find now the expressions and numerical values of $(\tilde{\theta}_-, \tilde{\theta}_0, \tilde{\theta}_+)$ by maximizing the likelihood obtained in **d.**.

[**f. – Extra credit**] *No need to show work.* Mark the correct answer.

- ☐ $l(\tilde{\theta}_-, \tilde{\theta}_0, \tilde{\theta}_+) < l(\theta_-, \theta_0, \theta_+)$ for all datasets
- ☐ $l(\tilde{\theta}_-, \tilde{\theta}_0, \tilde{\theta}_+) = l(\theta_-, \theta_0, \theta_+)$ for all datasets
- ☐ $l(\tilde{\theta}_-, \tilde{\theta}_0, \tilde{\theta}_+) > l(\theta_-, \theta_0, \theta_+)$ for all datasets
- ☐ None of the above.

Problem 3 – Properties of Mean and VarianceShow your work for all but a

X is a random variable with $E[X] = -1$, $Var(X) = 1$.

a. Y is another variable with $E[Y] = E[X]$, $Var(Y) = Var(X)$. Mark the correct answer to the questions below

$Y = X$ ☐ True ☐ False

$S_Y = S_X$ ☐ True ☐ False **b.** Let $Z = X + 1$. What is

Y is *Normal* $(-1, 1)$ ☐ True ☐ False
 $E[Z]$? What is $Var(Z)$?

c. Let $U = (X + 1)^2$. What is $E[U]$?

d. $W = X^2$. What is $E[W]$?

Problem 4 – ML estimation with missing information Show your work

You record n samples from a geometric distribution with parameter γ . However, due to a mistake, all that ends up being recorded is whether each sample was zero or non-zero. Geometric distribution $S = \{0, 1, 2, \dots, n, \dots\}$, density $f_\gamma(n) = (1 - \gamma)\gamma^n$, $\gamma \in (0, 1)$

a. What is the outcome space S of this experiment? (Find an appropriate symbol for “non-zero”.) What is the probability of each outcome in S ?

b. Write the log-likelihood of the data as a function of γ , using the probabilities you obtained in **a.** What are the sufficient statistics?

c. Now find the expression of maximum of the log-likelihood in **b.** and thus derive a formula for Maximum Likelihood estimate of γ .