STAT 391

3/11/25

Lecture 19

statistical decisions Ch. 15

Probabilistic Reasoning Ch. 14

Statistical decisions

[slides from '21]

$x \in S$ = outcome (sample) space

$a \in A$ actions $\Big\} \rightarrow P_a$ for $a \in A$

cost (x) (or utility)

$= c(x) \rightarrow$ random variable

Problem: knowing S, A, P_a for all $a \in A$, $c: S \rightarrow \mathbb{R}$

what is the best action?

Ex 1. How many servers?

$A = \{1, 2, \cdots M\}$ m = # servers

$\hookrightarrow$ or ∞

$S = \{0, 1, 2, \ldots \infty\}$ $x \in S$ = # requests / minute

(same every minute)

$P_X(x) = \text{Poisson}(\lambda) \leftarrow$ known $P_X(x) = \frac{\lambda^x}{x!} e^{-x}$

Costs Refuse request $R = 5$

Serving request $S = -10$ gain

Idle server $I = 1$

x requests $\Longrightarrow$ $x - m$ refused if $x \geq m$

m servers $\min(x, m)$ served

$m - x$ if $m > x$ idle

$$\Rightarrow c(x, m) = R \max(x - m, 0) + I \max(m - x, 0) + S \min(x, m)$$

random — not controlled (x)

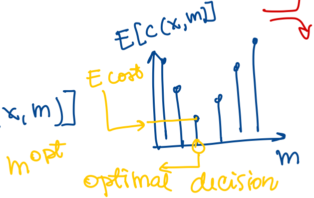
action (decision) (m)

Best m = $\underset{m \in A}{\arg\min}\ E_{P_X}[c(x, m)]$ Min expected cost

Stat

calculus

for $m \in A$, calculate $E[c(x, m)]$



if T (give test): $E[c] = 3 + 0 \cdot P(X,Y) + 100\, P(X,\bar{Y}) + 1 \cdot P(\bar{X},Y) + 0$

$\cong 3.3$

if $\bar{T}$ $\quad E[c] = \underbrace{P(X)}_{10^{-3}} \cdot 100 = 0.1$

$\Rightarrow \operatorname{argmin}_{T,\bar{T}} c = \bar{T}$

10^{-3} small prevalence !!

$10^{-1} \Leftarrow$ patient with symptoms -

Ex 2. Testing for HIV (screening)

$A = \{T, \bar{T}\}$

T: test

$S_X = \{X, \bar{X}\}$

X: HIV+, $\bar{X}$: HIV-

$S_Y = \{Y, \bar{Y}\}$ test result

Y: positive, $\bar{Y}$: negative

$P(Y|X)$ = detection probability $= 0.99$

$P(Y|\bar{X})$ = false alarm $= 0.03$

$P(\bar{Y}|\bar{X})$ = specificity $= 1 - 0.03$

$P(X) = 10^{-3}$

c	X	$\bar{X}$
Y	0	1
$\bar{Y}$	100	0

false alarm

mis detection

Cost $(T) = 3$

Other criteria for ~~statistical~~ decisions

Ex 3. Football $\to A = \{\text{run}, \text{kick}, \text{pass}\}$

run $\to$ P(down) - ??

kick $\to$ score, cost = -3

pass $\to$ ...

depend on
down $\in \{1,2,3,4\}$
distance to 10
current score
. . . .
distance to goal posts
. . .

$\to$ touchdown $= -5$ or -6

Best action $= \arg\min \cancel{E[\text{cost}]} = ?$

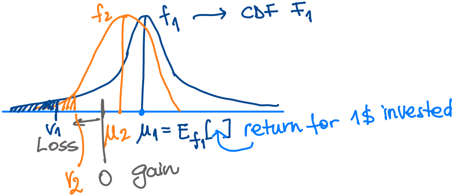
$\arg\max_{a \in A} P[\text{win}]$ optimistic

Criteria for making statistical decisions

- $\min_{a \in A} E[\text{cost}]$ (or $\max E[\text{utility}]$) ← many repetitions
- $\max_{a \in A} \Pr[\text{best outcome}]$ ← OPTIMIST
  - best outcome → an event - win game
  - $E^* = \{x \in S \mid \text{cost}(x) = \min\}$ best outcomes
- $\min_{a \in A} \Pr[\text{worst outcome}]$ ← Risk averse, cautious
- Value-at-Risk VaR

Investment strategy 1

— " — " — 2

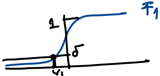


δ = risk level = 5%

$v_1 = F_1^{-1}(\delta) = VaR_\delta(f_1)$

$v_2 = F_2^{-1}(\delta) = VaR_\delta(f_2)$ ← dollars!! $\Rightarrow \Pr[\text{loss} \leq -VaR_\delta] = 95\% = 1-\delta$

$F_1(v_1) = \delta$



Gambling slot machine

pay \$1 ; win $x \sim f$

$E[x] \approx .99$

$E[\text{loss}] = -1 + E[\text{win}] = -0.01$

$VaR_{\delta=5\%} = -1 + 0 = -1$

$\Uparrow$ more than 5% of plays win 0

Extensions

$S = \{\text{outcomes}\}$

$A = \{\text{actions}\}$

$Z = \{\text{state}\}$

$P_{a,z}(x)$ outcome depends on state, action

cost(x) known

optimal a depends on z

$\pi(z) = \text{optimal decision}(z, P_{a,z}) \in A$

$a \in A$

policy can be computed (estimated, learned) in advance

· Classification

Problem: given z input, $y = \text{output} \in A$

$A = \{0,1\}$ binary classification

$A = \{1, 2, \ldots k\}$ multi-class classification

$a \in A$ classify z as $\underline{a}$

$\text{Cost} = \begin{cases} 0 & \text{if } a \text{ correct} \\ -1 & \text{if } a \text{ wrong} \end{cases}$

OR

$\text{cost} = \begin{cases} 0 & \text{if } a \text{ correct} \\ \text{cost}(a_{true}, a) & \text{if wrong} \end{cases}$

Criterion min $E[\text{cost}]$

- **Sequential decisions**
  - play games (backgammon)
    - a_1 = make a move (z_1)
    - state changes (opponent plays)
    - $z_2 \to$ move a_2
    - state changes
    - $z_3 \to$ move a_3 ← policy $\pi(z)$

Diagnosis

- observe patient
- $a_1 = A = \{$ test 1, test 2, X Ray, ..., diagnose disease B, ... $\}$
- → results come
- z = state of knowledge
- $a_2 \in \{$ tests, diagnose A, B, ..., treatments $\}$
- → more observations
- $a_3 \in \{ \ldots \}$

Markov Decision Process (MDP)

$z_1 \to z_2 \to z_3 \to z_4 \to$

state z_1, a_1 action; a_2; a_3

$P(z_2 | z_1, a_1)$

$P(\text{next } z \,|\, z_{current}, a)$

$\equiv P(z_{t+1} | z_t, a_t)$

$P(z_{t+1} | z_t, a(z_t))$

$= P(z_{t+1}$ ... ← depends only on z_t

policy

$\pi(z)$ = recommended (best) action in state z

- **Reinforcement Learning** = MDP with

1) P unknown

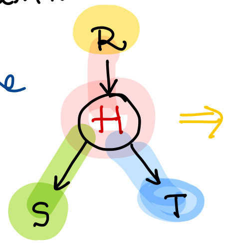
2) cost unknown

$\} \to$ P, cost (Reward), π ← learned (estimated) by repeated trials

Probabilistic Reasoning

Ex. Patient + doctor

Structure



$\Rightarrow$ $R \perp\!\!\!\perp T \mid H$

$R \perp\!\!\!\perp S \mid H$

$S \perp\!\!\!\perp T \mid H$

$\downarrow$

Bayesian network

directed, acyclic graph (DAG)

(no directed cycle)

para-meters of B.N. $\{ P_{X \mid \text{parents}(X)} \text{ for all } X \}$

unobserved

H = HIV infection $\in \{H, \bar{H}\}$ ← infected

T = HIV test $\in \{T, \bar{T}\}$ ← positive

observed

S = symptoms $\in \{S, \bar{S}\}$

R = risky behavior $\in \{R, \bar{R}\}$

$P(H \mid R) = 0.1$

$P(H \mid \bar{R}) = 10^{-3}$

$P(R) = 10^{-2}$ ← no parent

$P(S \mid H) = 0.8$

$P(S \mid \bar{H}) = 0.1$

$P(T \mid H) = .99$

$P(T \mid \bar{H}) = 0.03$ } $P_{T \mid \text{"parent"}}$

$P_{X \mid Y} = \dfrac{P_{XY}}{P_Y}$ ← joint, ← marginal

↑ conditional prob.

$P_{XY} = P_Y P_{X \mid Y}$ Chain Rule

$P_{Y \mid X} = \dfrac{P_Y P_{X \mid Y}}{P_X}$

Bayes' Rule

INDEPENDENCE

$X \perp\!\!\!\perp Y \Leftrightarrow$

$P_{XY} = P_X P_Y$

$\boxed{P_{X \mid Y} = P_X}$

$X \perp\!\!\!\perp Y \mid Z \Leftrightarrow$

$P_{XY \mid Z} = P_{X \mid Z} \cdot P_{Y \mid Z}$

$\boxed{P_{X \mid YZ} = P_{X \mid Z}}$

X conditionally indep of Y given Z

$S = S_R \times S_H \times S_S \times S_T = \{ (r, h, s, t),\ r \in \{R, \bar{R}\},\ h \in \{H, \bar{H}\} \dots \}$

sample space

$|S| = 2^4$

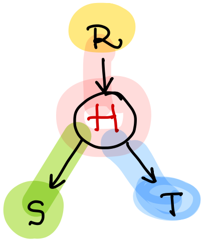
$P_{\underbrace{R}_{Y}\underbrace{HST}_{X}}(r, h, s, t) = P_R(r)\, P_{HST \mid R}(h, s, t \mid r)$

$= P_R(r)\, P_{H \mid R}(h \mid r)\, P_{ST \mid HR}(s, t, h, r)$

$= P_R(r)\, P_{H \mid R}(h, r)\, P_{S \mid H\cancel{R}}(s \mid h, r)\, P_{T \mid \cancel{SHR}}(t \mid s, h, r)$

$P_{T \mid H}$

$P_{T \mid HR}$



Query: $P_{H|R,S,T} = ?$

$r = R$
$s = S$
$t = T$

$$P_{H|R,S,T} = \frac{P_{RHST}}{P_{RST}} = \frac{P_R P_{H|R} P_{S|H} P_{T|H}}{\ldots\ldots} =$$

$P_{RST} = P_{RHST}(R,H,S,T) + P_{RHST}(R,\bar{H},S,T) \leftarrow$ Total Prob

$$P_{H|R,S} = \frac{P_{RHS}}{P_{R,S}}$$

Rule 1 numerator = 1 term , denominator = = sum

Rule 2 $\rightarrow$ keep them $\exists$ table

$\leftarrow$ use Bayes Rule

T.B. CONTINUED ...