Appendix to Lecture Notes 0 – The bias of $\hat{\sigma}^2$

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$$
\begin{aligned}
\mathbb{E}[\hat{\sigma}^2] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\hat{\mu})^2\right] && (1)\\
&= \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n}(X_i^2-2\hat{\mu}X_i+\hat{\mu}^2)\right] && (2)\\
&= \frac{1}{n}\left\{\sum_{i=1}^{n}\underbrace{\mathbb{E}[X_i^2]}_{\sigma^2+\mu^2}+\mathbb{E}[-2\hat{\mu}\underbrace{\sum_{i=1}^{n}X_i}_{n\hat{\mu}}+n\hat{\mu}^2]\right\} && (3)\\
&= \frac{1}{n}\left\{n(\sigma^2+\mu^2)+\mathbb{E}[-2n\hat{\mu}^2+n\hat{\mu}^2]\right\} && (4)\\
&= \sigma^2+\mu^2-\mathbb{E}[\hat{\mu}^2] && (5)\\
&= \sigma^2+\mu^2-(\mathbb{E}[\hat{\mu}]^2+Var\hat{\mu}) && (6)\\
&= \sigma^2+\mu^2-(\mu^2+\sigma^2/n) && (7)\\
&= \sigma^2-\frac{1}{n}\sigma^2 && (8)
\end{aligned}
$$

This derivation used the *second moment* formula $\mathbb{E}[Z^2] = VarZ + \mathbb{E}[Z]^2$.